Examining the Uncertainty-Investment Relationship under Various Stochastic Processes

George Yungchih Wang

National Kaohsiung University of Applied Sciences, Taiwan

Conventional belief in a negative relationship between uncertainty and investment has dominated real options theory for a long time. These studies argue that increased uncertainty causes a decrease in the current level of investments by raising optimal investment trigger. This paper postulates an argument that increased uncertainty, in certain situations, may actually encourage investment. Since earlier studies mostly base their arguments on the assumption of geometric Brownian motion, the study extends the assumption to alternative stochastic processes, such as mixed diffusion-jump, mean-reverting process, and jump amplitude process. To investigate the relationship between uncertainty and investment, a general approach of Monte Carlo simulation is developed to derive optimal investment trigger under various stochastic processes when the closed-form solution could not be readily obtained. The overall effect of uncertainty on investment may be interpreted by the probability of investing, and the main result finds that the relationship appears to be an inverted U-shaped curve between uncertainty and investment. The implication is that uncertainty does not always discourage investment even under several sources of uncertainty. Therefore, high-risk projects are not always dominated by low-risk projects because high-risk projects may induce investment due to the higher probability of investing.

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1. Introduction

The relationship between uncertainty and investment has fascinated financial economists for a long time. Early literature on real options theory argues that increased uncertainty causes a decrease in the current level of investment by raising the value of option of waiting. For example, Cukierman (1980) presents a Bayesian framework to address the idea that an investment opportunity can be more valuable by waiting longer for more information arrivals. Pindyck (1988 and 1991) and Dixit (1989 and 1992) also find that a higher level of uncertainty not only increases option value, but also brings about a higher optimal investment trigger[1] to such an extent that uncertainty may in effect discourage investment. Abel and Eberly (1999) further contend that the uncertainty-investment relationship is positive for a lower of uncertainty while the relationship is negative for a high level of uncertainty, suggesting an inverted U-shaped relationship.

Extending standard real options theory, Sarkar (2000) and Rhys, Song, and Jindrichovska (2002) explore the relationship between uncertainty and investment by asking the question how much the likelihood is that a project value, V, would reach optimal investment trigger, V^*, given that the project value evolves as a geometric Brownian motion (GBM). Both studies apply a similar probability function, and find that the uncertainty-investment relationship is not always negative. They show that when investment uncertainty follows a GBM process, increased uncertainty, in certain situations, may encourage investment due to a higher probability of investing or an earlier time of first passage. Recent studies on investment theory suggest that the relationship between uncertainty and investment mostly is nonlinear. Lensink and Murinde (2006) empirically examine

[1] An optimal investment trigger is defined as a threshold value to launch a project when the present value of cash flows generated from an investment project is greater than the threshold value.

the data of UK firms and propose the inverted-U hypothesis for the effect of uncertainty on investment. In addition, Wong (2007) analyzes optimal investment timing in a real options model and argues that optimal investment trigger exhibits a U-shaped pattern against project volatility.

While a GBM process, also known as a diffusion process, could describe the nature of random walk, it fails to capture the impact of stochastic informational arrivals. Investment literature has suggested alternative stochastic processes to cope with the problem. The first relaxation of GBM was suggested by Merton (1976) and later Trigeorgis (1990), who propose a mixed diffusion-jump process in order to describe both incremental changes over time and Poisson jumps. On the other hand, Pennings and Lint (1997) also advocate a jump amplitude process to explain an underlying variable with stochastic jump direction and jump size.

The GBM assumption not only fails to describe the jump property, but also could not capture the nature of mean reversion in commodity prices. Metcalf and Hassett (1995) argue that a GBM process is fundamentally inappropriate if the underlying asset of an investment project has the mean-reverting nature. The property of mean reversion especially exists in the price behaviors of commodities and natural resources.

Based on the preceding discussion, when the underlying process is not a GBM, the relationship between uncertainty and investment is generally non-monotonic.[2] This paper aims to investigate the uncertainty-investment relationship by relaxing the assumption of state variable to various stochastic processes with the technique of Monte Carlo simulation. The stochastic processes of interest are GBM, mixed diffusion-jump (MX), mean-reverting process (MR), and jump amplitude process (JA). Earlier studies, such as Sarkar (2000) and Rhys, Song, and Jindrichovska (2002), apply a probability function to measure the probability of reaching a critical value under a GBM, yet their models fail to address the relationship between uncertainty and investment under an alternative stochastic process. Literature has indicated that when the state variable follows an alternative process, there is generally no closed-form solution for the optimal investment trigger and the resulting function of investment. In this situation, Monte Carlo simulation is suggested in exploring the relationship between uncertainty and investment due to its convenience to compute and flexibility to handle.

The rest of the paper is organized as follows: Section 2 introduces the specifications of alternative stochastic processes both in continuous time and in discrete time, serving as a foundation for the subsequent sections. Section 3 proposes the approach of Monte Carlo simulation for deriving optimal investment trigger in a more general setting. Section 4 examines the relationship between uncertainty and investment by decomposing the overall effect into the effect of uncertainty and the effect of realization. The probability of investing is then suggested to measure the overall effect of uncertainty on investment. Section 5 gives concluding remarks.

2. Optimal Investment Trigger

Since volatility component in stochastic process is regarded as a major source of uncertainty in evaluating capital investments, in this section a variety of stochastic processes are introduced as well as the derivation of optimal investment triggers. A framework of Monte Carlo simulation for deriving optimal investment is then proposed for alternative stochastic models that could not be readily solved for a closed-form solution.

2.1 Geometric Brownian Motion

In traditional real options literature, the GBM assumption is widely assumed to address for the uncertainty of random walk with a drift rate and a random term. The main property of GBM is that the rate of return is assumed to be normally distributed, implying a lognormal distribution of the project value. A GBM in continuous time is expressed as follows:

$$dV = \alpha V dt + \sigma V dz \tag{1}$$

[2] See a more thorough survey in Dixit and Pindyck (1994).

where α, σ, and *dz* denote drift rate, instantaneous volatility, and an increment of a standard Wiener process, respectively.

A GBM process in discrete time could be changed into the following form:

$$\Delta \ln V = v\Delta t + \sigma\sqrt{\Delta t}\varepsilon \tag{2}$$

where Δt and ε represent a small interval of time and a random drawing from a standard normal distribution, respectively, and $v=\alpha-\sigma^2/2$.[3]

Suppose a firm is presented with an investment opportunity that pays an irreversible investment cost, *I*, in return for an uncertain project value, *V*. This is a standard problem of optimal investment timing in real options literature. *V* is considered to be the major source of uncertainty and is normally assumed to follow a GBM as in Equation (1) due to the ease of deriving a tractable solution. The value of an investment opportunity is determined by an optimal investment policy that maximizes the option value. Let *F(V)* denote the value of the investment opportunity and the superscript * denote optimality. McDonald and Siegel (1986), Pindyck (1991), and Dixit and Pindyck (1994) have demonstrated that the optimal investment trigger is given by

$$V^*_{GBM} = \left(\frac{b_1}{b_1 - 1}\right) I \tag{3}$$

where V^*_{GBM} and *I* denote the optimal GBM trigger and the investment cost, respectively, and

$$b_1 = \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right) + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{4}$$

where δ represents convenience yield of holding a project, which also implies the opportunity cost of deferring a project.

2.2 Mixed Diffusion-Jump Process

While a GBM process could describe the incremental changes of random walk, the process fails to capture the significant impact of random informational arrival. In reality, informational arrivals could be market collapse, company bankruptcy, or unexpected natural distress. A mixed diffusion-jump process is thus proposed to include a Poisson jump into a GBM process. There are a variety of forms of a mixed diffusion-jump process, one of which is proposed by Merton (1976) in the financial option pricing problem and then applied by Trigeorgis (1990) in the context of evaluating an investment opportunity with competitive arrivals. A mixed diffusion-jump process in continuous time is expressed as follows:

$$dV = (\alpha - \lambda k)Vdt + \sigma V dz + V dq_1 \tag{5}$$

where dq_1 is an increment of a Poisson jump process with a mean arrival rate λ such that

$$dq_1 = \begin{cases} \varphi & \text{with a probability of } \lambda dt \\ 0 & \text{with a probability of } 1 - \lambda dt \end{cases} \tag{6}$$

where $\varphi \sim N(k, \sigma_\varphi)$ denotes a proportional jump relative to *V* if a jump occurs.

Note that the Poisson jump term dq_1 is assumed to be independent of *dz* such that $E(dq_1 dz)=0$. Equation (6) also reveals that the actual growth rate of such a mixed diffusion-jump process is not *a* but instead $(a-\lambda k)$ in order to adjust the influence of a Poisson event. For the simulation purpose, the

3 Since GBM is log-normally distributed, a more explicit form of Equation (2) is given below: $V_{t+\Delta t} = V_t \left[e^{(v\Delta t + \sigma\sqrt{\Delta t}\varepsilon_t)} \right]$

discrete-time version of the mixed diffusion-jump process is given as follows:

$$\Delta \ln V = v\Delta t + \sigma\sqrt{\Delta t}\varepsilon + D_1 \tag{7}$$

where D_1 denotes an increment of a Poisson jump in discrete time with a mean arrival rate λ such that

$$D_1 = \begin{cases} \varphi & \text{with a probability of } \lambda\Delta t \\ 0 & \text{with a probability of } 1 - \lambda\Delta t \end{cases} \tag{8}$$

It is worth noting that McDonald and Siegel (1986) and Dixit and Pindyck (1994) also propose a mixed diffusion-jump process with the sign of the jump term changed into negative to describe the situation in that the project becomes suddenly worthless when a major competitor of the same product enters the market. For a graphical comparison, Figure 1 exhibits two simulated sample paths of a mixed diffusion-jump process with the GBM as a comparison.

Figure 1
A Graphical Comparison of GBM and Mixed Diffusion-Jump

Note: $V_0 = 100,\ \alpha = 5\%,\ \sigma = 20\%,\ \Delta t = 1/52,\ T = 4,\ \lambda = 2\ (Mixed1)$

For an investment opportunity whose value follows a mixed diffusion-jump process, McDonald and Siegel (1986) and Dixit and Pindyck (1994) show that when the value of the project may be appropriated by competitive arrivals such that the project becomes suddenly worthless, the solution of optimal trigger under such a mixed diffusion-jump process, V^*_{MX}, has the same form as Equation (3) with b_1 substituted by b_2 as follows:

$$b_2 = \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right) + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} \tag{9}$$

where λ denotes the jump intensity of competitive arrivals.

2.3 Mean-Reverting Process

In addition to a GBM process, a mean-reverting process is proposed to describe uncertainty in which the state variable evolves according to a long-run mean. According to Schwartz (1997), a mean-reverting process is commonly seen in the macroeconomic variables, e.g., interest rate, and commodity prices, e.g., metal and oil. One unique property of a mean-reverting process is that its growth rate is not a constant but instead a function of a difference between current value and long-run mean, suggesting that the growth rate in effect responds to disequilibrium. Dixit and Pindyck (1994, Ch. 5) examine the value of an investment opportunity whose value follows a mean-reverting process. The specification of this commonly used mean-reverting process is given below:

$$dV = \eta\left(\bar{V} - V\right)Vdt + \sigma V dz \tag{10}$$

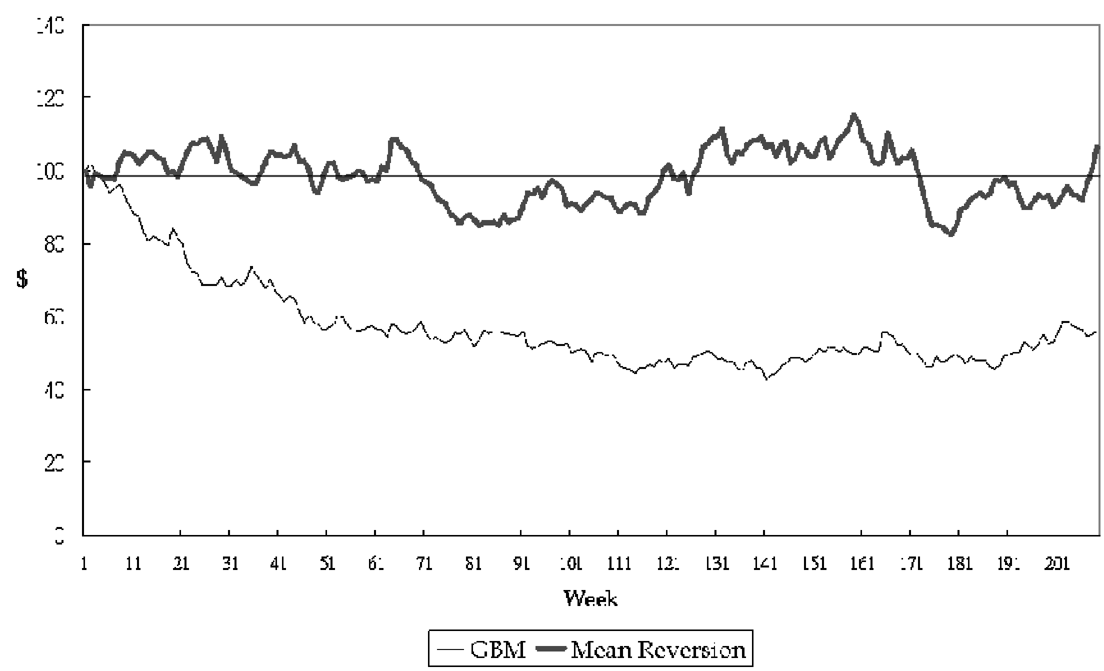
where η denotes a speed of mean reversion and $\bar{V}$ is a long-run mean.

As there are many ways to specify a mean-reverting process, Dixit and Pindyck's specification is somewhat arbitrary but convenient to find a "quasi-analytical" solution for the value of the project. Equation (10) can be alternatively expressed into the following equation in discrete time:

$$\Delta \ln V = \left[\eta\left(\bar{V} - V\right) - \frac{1}{2}\sigma^2\right]\Delta t + \sigma\sqrt{\Delta t}\varepsilon \tag{11}$$

For a graphical comparison, we give a simulated sample path of mean reversion according to Equation (11) with the GBM as a comparison in Figure 2.

Figure 2
A Graphical Comparison of Mean Reversion and GBM



Note: $V_0 = \bar{V} = 100,\ \alpha = 5\%,\ \sigma = 20\%,\ \Delta t = 1/52,\ T = 4,\ \eta = 0.03$

Under the assumption of a mean-reverting process, Dixit and Pindyck (1994) provide the solutions of an investment opportunity and optimal investment trigger, respectively, as follows:

$$F(V) = BV^{\theta}G(x;\theta,g) \tag{12}$$

$$V_{MR}^{*} = F\left(V_{MR}^{*}\right) + I \tag{13}$$

where $$\theta = \frac{1}{2} - \frac{\eta\bar{V}}{\sigma^2} + \sqrt{\left[\frac{1}{2} - \frac{\eta\bar{V}}{\sigma^2}\right]^2 + \frac{2r}{\sigma^2}} \tag{14}$$

$$x = \frac{2\eta}{\sigma^2}V \tag{15}$$

$$g = 2\theta + \frac{2\eta\bar{V}}{\sigma^2}$$

$$G(x;\theta,g) = 1 + \frac{\theta}{g}x + \frac{\theta(\theta+1)}{g(g+1)}\frac{x^2}{2!} + \frac{\theta(\theta+1)(\theta+2)}{g(g+1)(g+2)}\frac{x^3}{3!} + \cdots \tag{16}$$

Note that $G(x,\theta,g)$ is an infinite confluent hypergeometric function, and thus the value of the investment opportunity cannot be readily solved. Both Equations (12) and (13) must be solved numerically from an iterative procedure to obtain V^* and $F(V^*)$.

2.4 Jump Amplitude Process

Pennings and Lint (1997) argue that a GBM fails to capture the major impact of technological breakthrough and informational arrivals in an R&D project, thus suggesting a jump amplitude process to evaluate such an investment opportunity. A jump amplitude process differs from the other types of jump processes in a sense that it allows for a random jump direction and a stochastic jump size in order to characterize the nature of R&D investments. A jump amplitude process can be mathematically expressed as follows:

$$dV = \alpha V dt + V dq_2 \tag{17}$$

where dq_2 an increment of a stochastic jump process. The jump term, dq_2, is characterized by a parameter of jump intensity λ such that

$$dq_2 = \begin{cases} \varphi & \text{with a probability of } \lambda dt \\ 0 & \text{with a probability of } 1 - \lambda dt \end{cases} \tag{18}$$

where φ denotes a proportional jump relative to V.

By definition, $\varphi = X\Gamma$ where $X=1$ or -1, $P(X=1)=p$, and $\Gamma / X \sim Wei(\gamma_X, 2)$. The jump amplitude process in discrete time is modeled as follows:

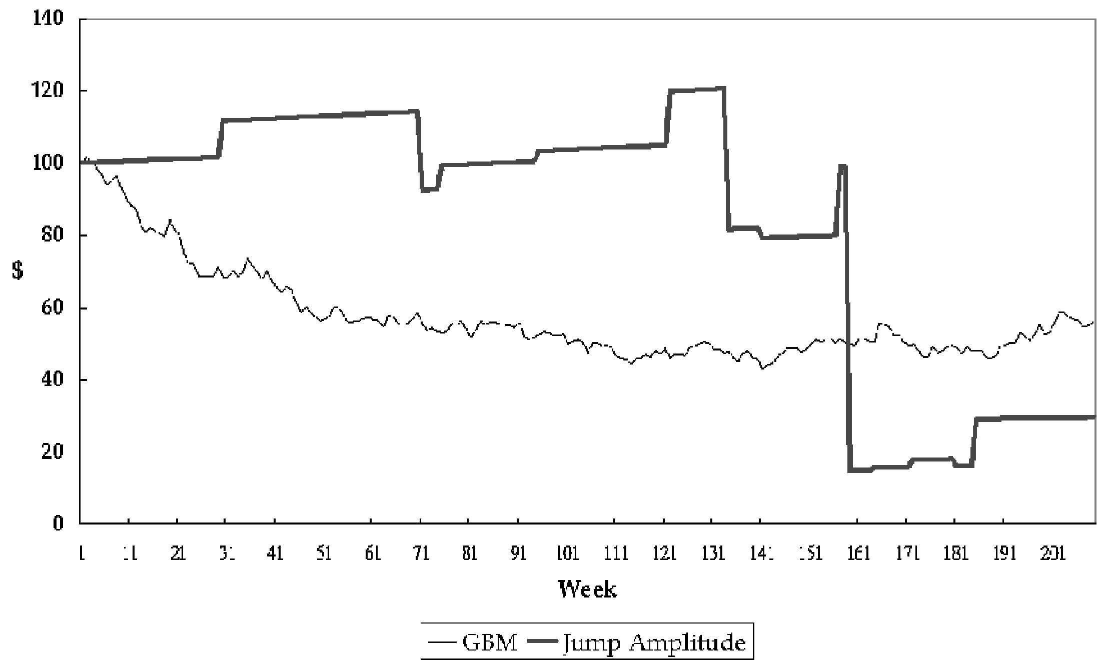
$$\Delta \ln V = v\Delta t + D_2 \tag{19}$$

where D_2 denotes an increment of a stochastic jump component in discrete time with a mean arrival rate λ, and D_2 is expressed by

$$D_2 = \begin{cases} \varphi & \text{with a probability of } \lambda \Delta t \\ 0 & \text{with a probability of } 1 - \lambda \Delta t \end{cases} \tag{20}$$

Since a jump amplitude process allows both positive and negative jumps, the estimation of the probability of up-jumps and down-jumps is important in specifying the process. Figure 3 presents a simulated jump amplitude process, by assuming $P(X=1)=0.5$, i.e., a 50-50 chance for an up-jump and a down-jump. As shown in Figure 3, a jump amplitude process evolves in a very different way compared to a GBM.

Figure 3
A Graphical Comparison of GBM and Jump Amplitude



Note: $V_0 = 100,\ \alpha = 5\%,\ \sigma = 20\%,\ \Delta t = 1/52,\ T = 4,\ \lambda = 2,\ p = 50\%,\ \gamma_X = 0.1$

Since there is no closed-form solution for an investment opportunity whose uncertainty evolves as a jump amplitude process, numerical techniques must be applied to solve both *F(V*)* and *V**. (Pennings and Lint, 1997) Monte Carlo simulation therefore is introduced in the next section for this purpose.

3. The Procedure of Monte Carlo Simulation

Since uncertainty under an alternative stochastic process complicates capital investments in that closed form solutions for optimal investment triggers no longer exist, numerical techniques must be applied. In the section, Monte Carlo simulation in discrete time is first presented to derive optimal investment triggers. An iterative procedure is then proposed for computation convenience. Finally, we define the function of probability of investing to examine the relationship between uncertainty and investment. It is important to note that the procedure to examine the uncertainty-investment relationship is independent of stochastic process underlying the state variable.

3.1 Deriving Optimal Investment Triggers

Suppose that the investment opportunity will disappear at a finite future time T, if the firm does not take any actions. Therefore, the value of an investment opportunity at time T, given the information set ψ_T, is expressed as follows:

$$F_T(V_T) = \max(V_T - I, 0)|\phi_T \tag{21}$$

According to Equation (21), the value of investment opportunity at time t can be given by

$$F_t = E^P\left[e^{-(T-t)\rho}\max\left(V_T - I, 0\right)\right] \tag{22}$$

where E^P denotes an expectation operator in a risk-adjusted world, P a risk-adjusted probability measure, and ρ a risk-adjusted discount rate.

In the risk-neutral world, F_t can be derived from

$$F_t = E^Q\left[e^{-(T-t)r}\max\left(V_T - I, 0\right)\right] \tag{23}$$

or

$$F_t = e^{-(T-t)r}E^Q\left[\max\left(V_T - I, 0\right)\right] \tag{24}$$

where r denotes a risk-free rate and Q a risk-neutral probability measure.

Note that when the market is complete or the investor is risk-neutral, there exists a unique risk-neutral probability measure Q such that F can be evaluated by Equation (23). If the market is incomplete or the investor is risk-averse, there does not exist such a unique Q and thus F can be evaluated by Equation (22). Equations (23) or (24) states a fundamental equation for valuing an investment opportunity in a numerical procedure of Monte Carlo simulation, given any V_t.

To determine the optimal investment trigger, we need to search for an investment trigger V^*_t such that the net present value of taking the project, V^*_t-I, can compensate the loss of option of waiting, $F_t(V^*_t)$. This optimal investment policy can be described by a value-matching condition as follows:

$$F_t\left(V_t^*\right) = V_t^* - I \tag{25}$$

or alternatively

$$V_t^* = F_t\left(V_t^*\right) + I \tag{26}$$

To rule out the possibility of an arbitrage opportunity or the "kinked" situation[4], the first derivative of the value-matching condition with respect to the state variable at the maximum must be equal on both sides. This is the famous Samuelson smooth-pasting condition given below:

$$F_{V_t^*}\left(V_t^*\right) = 1 \tag{27}$$

By substituting Equation (24) into (26), we have optimal investment trigger, V^*_t, expressed as follows:

$$V_t^* = e^{-(T-t)r}E^Q\left[\max\left(V_T - I, 0\right)\right]\Big|_{V_t = V_t^*} + I \tag{28}$$

Equations (27) and (28) represents two fundamental equations necessary to derive optimal investment trigger, V^*_t. There are two major advantages of applying the approach to derive optimal investment triggers. First, Equations (27) and (28) would hold regardless of the underlying assumption of stochastic process. As mentioned earlier, literature has indicated that there is a closed-form solution for optimal investment triggers under a GBM (McDonald and Siegel, 1986; Pindyck 1991; Dixit and Pindyck, 1994). For the projects whose state variable follows an alternative process, the closed-form solutions for optimal investment triggers are generally unavailable. Therefore, the proposed approach is particularly advantageous when the project value follows an alternative stochastic process.

[4] See Dixit and Pindyck (1994, Ch. 4).

Second, Equation (28) can be conveniently applied to the case that the investment opportunity will disappear in a known expiration of time in future. Conventional real options literature mostly makes an implicit assumption that the investment opportunity can exist in an infinite time horizon for the convenience in deriving analytical solutions. This assumption is not quite realistic in practice, especially when the factor of technology obsolesce is involved with the project or the deferral option has an expiration date.

It is important to note that growth rate (or drift rate) must be assumed to be less than discount rate (either risk-adjusted discount rate or risk-free rate), otherwise it will be never optimal to early exercise an investment opportunity before the expiration time. By setting growth rate less than discount rate, it is equivalent to assume that there exists a positive convenience yield which accounts for an opportunity cost (denoted by δ) of delaying the construction of a project. In a risk-neutral world, when a convenience yield plays a role in real options valuation, the actual growth rate of an underlying process must be adjusted by reducing an amount of convenience yield.[5] Therefore, as the opportunity cost of delaying a project becomes larger, the actual growth rate of the underlying process becomes smaller.

Since the approach is based on the valuation of a European-style option, one may ask whether the early exercise premium matters in real options with the American characteristic. According to Barone-Adesi and Whaley (1987), for an at-the-money option with a moderate opportunity cost ($\delta = 4\%$) and a short time horizon ($T = 0.25$ or 0.5), the early exercise premium is estimated to be 0.00%.[6] For an at-the-money option with a longer time horizon (T=2), the early exercise premium is estimated to be less than 1%.[7] Therefore, it is practical to assume that the effect of early exercise premiums is minimal and may be negligible in the situations where the at-the-money project is of interest.

3.2 Iterative Procedure

As mentioned in the preceding subsection, the technique of Monte Carlo simulation can be applied to derive optimal investment trigger under any stochastic process. Following the idea, we then describe the algorithm of an iterative procedure in the implementation of Monte Carlo simulation. As the first step of the procedure, a large number of random paths are generated according to a specific stochastic process in order to compute terminal payoffs. The next step is to discount terminal payoffs backward at a discount rate, which equals the risk-free rate in the risk-neutral world or a risk-adjusted rate in the risk-adjusted world. If the discount rate is not certain over the investment horizon, an interest rate process needs to be simulated simultaneously. For a reasonable short time horizon, we can assume that the discount rate is constant for simplicity. The value of an investment opportunity can be computed from the mean of the discounted payoffs. The value of optimal investment trigger must be derived from an iterative procedure which equates V* and F(V*)+I.

To derive the optimal investment trigger, V*, in the iterative procedure it is necessary to start with the first two initial values V_1 and V_2, where V_1 and V_2 are two guessed numbers which are lower than V*. Next, V_1 and V_2 are then applied to evaluate the right-hand side of Equation (28). Since it is very unlikely that any of the two numbers would equate the value-matching condition, we then compute the slope (χ) of the line connecting both numbers as follows:

$$\chi = \frac{F(V_2) - F(V_1)}{V_2 - V_1} \tag{29}$$

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[5] If the world is assumed to be risk-neutral, the discount rate in the framework becomes risk-free rate. Thus, the actual drift in the stochastic process equals risk-free rate less convenience yield. This treatment is called "equivalent risk-neutral valuation". See the discussions in Dixit and Pindyck (1994, p.121-125).

[6] The risk-free rate is assumed to be 8%. Refer to Table 2 in Barone-Adesi and Whaley (1987).

[7] Refer to Table 5 in Barone-Adesi and Whaley (1987).

Suppose there is a larger number V_3, i.e., $V_3 > V_2 > V_1$, such that the following relationship holds:

$$V_3 = \left[F(V_2) + I\right] + \chi(V_3 - V_2) \tag{30}$$

Equation (30) could be rearranged for V_3 as shown below:

$$V_3 = \frac{\left[F(V_2) + I\right] - \chi V_2}{1 - \chi} \tag{31}$$

Now V_3 can be used to compute a new option value, $F(V_3)$, and a new slope with respect to V_2. Since it is very unlikely that both sides of the value-matching condition are exactly equal during the iteration, an acceptance criterion must be established in the iterative procedure. Let ε be the level of acceptance tolerance. Thus, the new slope, according to Equation (27), must satisfy the following criterion to stop the iterative procedure:

$$\left|\chi - 1\right| \le \varepsilon \tag{32}$$

Note that as a smallerεis chosen, the longer the iterative procedure it takes, and vice versa.

3.3 Probability of Investing

Investment literature has suggested that the optimal investment rule to launch a project is at the time when the value of the project exceeds the optimal investment trigger. This also means that the optimal investment timing is decided not only when the investment cost and the option value are covered, but also the value of the investment opportunity is maximized. Since option pricing theory suggests that an increase in uncertainty raises the option value, some researchers therefore argue that uncertainty may in effect discourage investment.[8] Recent study on the relationship between uncertainty and investment (Sarkar, 2000; Lensink and Murinde, 2007) suggest the nonlinear relationship between uncertainty and investment. Lensink and Murinde (2007) examine the UK evidence and find that the effect of uncertainty on corporate investment could be approximated by an inverted-U shaped relationship, meaning that at low levels of uncertainty the effect is positive, but it becomes negative at high levels of uncertainty. Following the earlier studies in the line of the uncertainty-investment relationship, we extend the idea to reexamine the overall effects of uncertainty on investment given that the state variable follows an alternative stochastic process.

Sarkar (2000) suggests a probability function of reaching optimal investment trigger to compute the probability of investing under a GBM process. Since the probability function of investing is unavailable in the situation in which the underlying variable follows an alternative stochastic process, Monte Carlo simulation is suggested for a more general purpose to measure the probability of simulated random paths reaching optimal investment trigger.

The procedure of Monte Carlo technique begins with simulating a large number of sample paths, given a particular stochastic process. The stochastic processes under consideration are GBM, mixed diffusion-jump process, mean-reverting process, and jump amplitude process, which are simulated according to Equations (2), (7), (11), and (19), respectively. It is worth noting that the actual drift rate of a simulated stochastic process must be reduced by a convenience yield, i.e., an opportunity cost of holding a project.

The optimal investment trigger under a given stochastic process can be derived from the approach described in the preceding section. In each simulation trial, if at any time the project value V_t is greater than V^*, this simulation trial is counted as a case of taking on the project. To examine the overall effect of uncertainty under a specific stochastic process on investment, the probability of

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[8] Cukierman (1980), Caballero (1991), Mauer and Ott (1995), and Metcalf and Hassett (1995).

investing is then measured by computing the total cases of taking on the project out of the total simulation trials. The total number of simulation trials should be large enough to ensure a robust result. Thus, a higher probability of investing implies a greater chance of project acceptance, hence a positive impact on investment, and vice versa.

$$P(Inv) = P\left(V_t > V^*\right)\big|_{n \text{ trials}} = \frac{k}{n} \tag{33}$$

where n is the number of total simulation trials and k is the total cases of taking on the project.

On the relationship between uncertainty and investment, we argue that there are two opposing forces within the overall effect of uncertainty on investment. The first force is termed the "variance effect", which states that an increase in instantaneous volatility would raise the level of optimal investment trigger and therefore delay investment. The variance effect could be identified by observing how optimal investment trigger changes as project volatility changes. The second force is called the "realization effect", which describes the likelihood of reaching optimal investment trigger may increase due to a higher level of instantaneous volatility. The realization effect could be identified by observing how the probability of investing changes with increased volatility. Consequently, the relationship between uncertainty and investment could be obtained by combining these two effects.

4. Numerical Analysis for Examining the Uncertainty-Investment Relationship

To illustrate the relationship between uncertainty and investment, numerical analysis is conducted on a base case. Since the options to invest matter most in investment decisions when an investment project is at the money, the base case is constructed in the situation of zero NPV. (Copeland and Antikarov, 2001) Consider an investment project whose investment cost, I, is 100 in return for a project value at time t, V_t. To construct an at-the-money case, V_0 is assumed to be same as I, 100. Since we are interested in capital investment, project duration is assumed to be 5, 10, or 15 years. Time interval is assumed to be $\triangle t=1/52$ in order to create a similarly continuous path. The risk-free rate (r) is set to be 5%. Based on the base case, four stochastic processes of interest are assumed, followed by applying the procedure of Monte Carlo simulation to investigate the uncertainty-investment relationship

4.1 The Uncertainty-Investment Relationship under a GBM

In the case that the underlying stochastic process follows a GBM, the variance effect can be readily observed from the changes in optimal investment triggers as the project volatility changes. With the parameter values in the base case, the variance effect under a GBM is exhibited in Figure 4.

As displayed in Figure 4, it is obvious that the optimal investment trigger, V^*_{GBM}, increases with σ, and decreases with δ. The intuition underlying the positive relationship of V^*_{GBM} andσ is that as investment triggers increases with uncertainty, management should defer the project longer until the market condition becomes favorable, i.e., $V_t>V^*$. However, as the opportunity cost of holding a project increases, it then becomes insensible to postpone the project any longer, hence lowering optimal investment triggers.

To identify the realization effect, 10,000 random paths are simulated to evaluate the probability of investing. The simulation result is displayed in Figure 5.

As seen in Figure 5, the probability of investing is initially an increasing function of volatility, but after a certain point it becomes a decreasing function of volatility. This means for a lower level of volatility, an increase in uncertainty actually raises the probability of investing and thus has a positive influence on investment, while an increase in uncertainty, on the other hand, discourages investment for a higher level of volatility. This result of the inverted U relationship between uncertainty and investment is consistent with the finding in Lensink and Murinde (2007).

Figure 4
The Variance Effect under a GBM

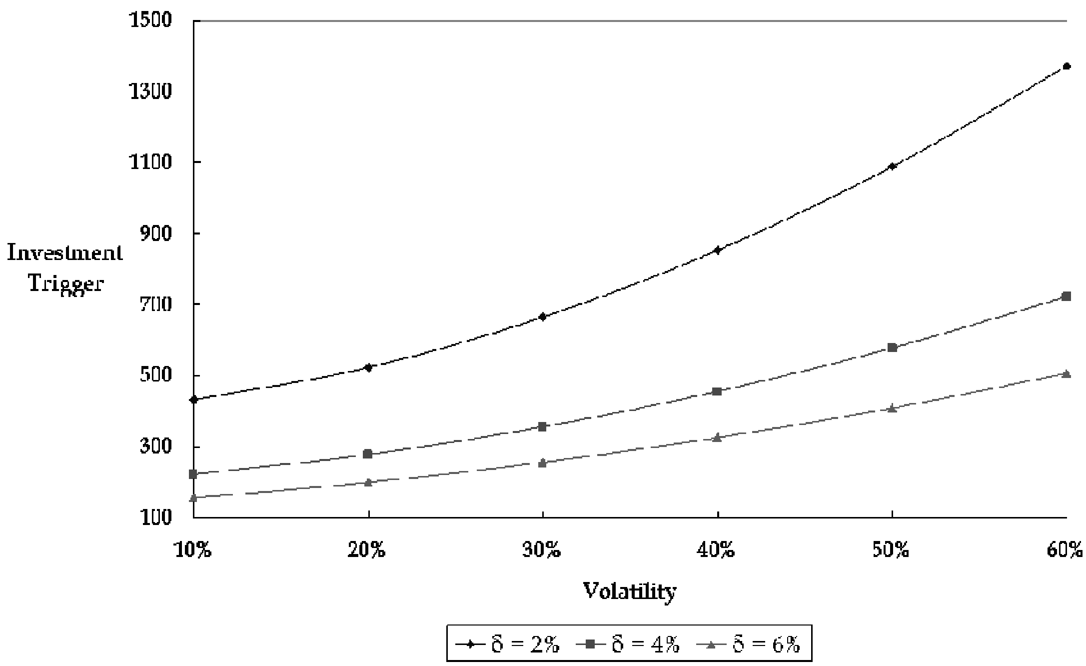
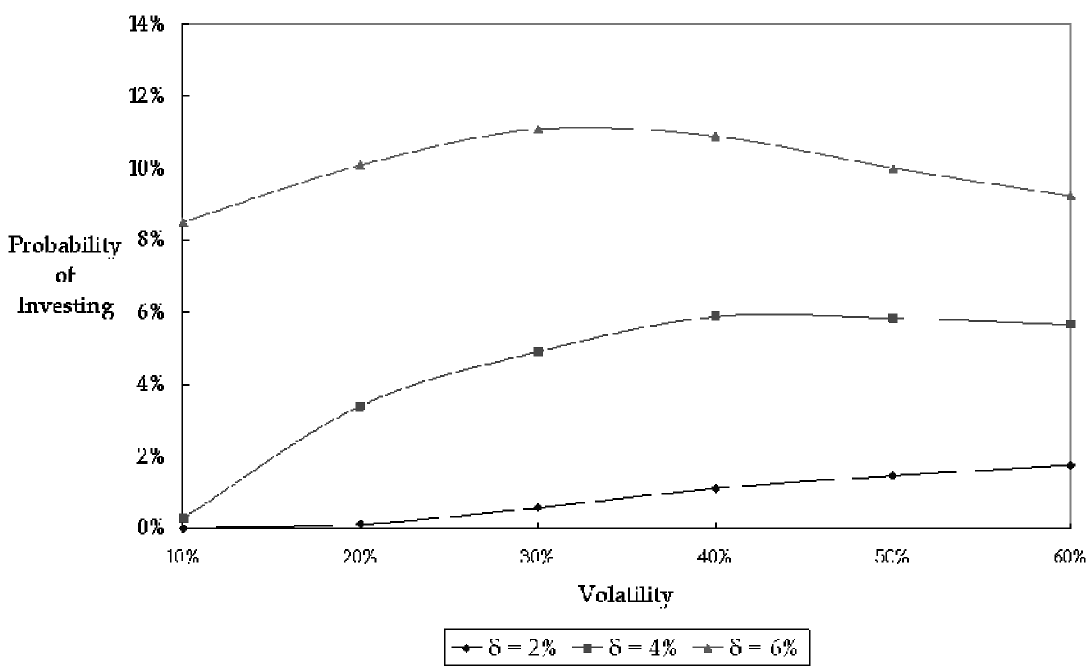


Figure 5
The Probability of Investing as a Function of Volatility (σ) Given a GBM Process



In addition, the probability of investing, as shown in Figure 5, increases with the opportunity cost of holding a project, given volatility being unchanged. Thus, an increased convenience yield may have a positive impact on investment, encouraging management to launch investment sooner.

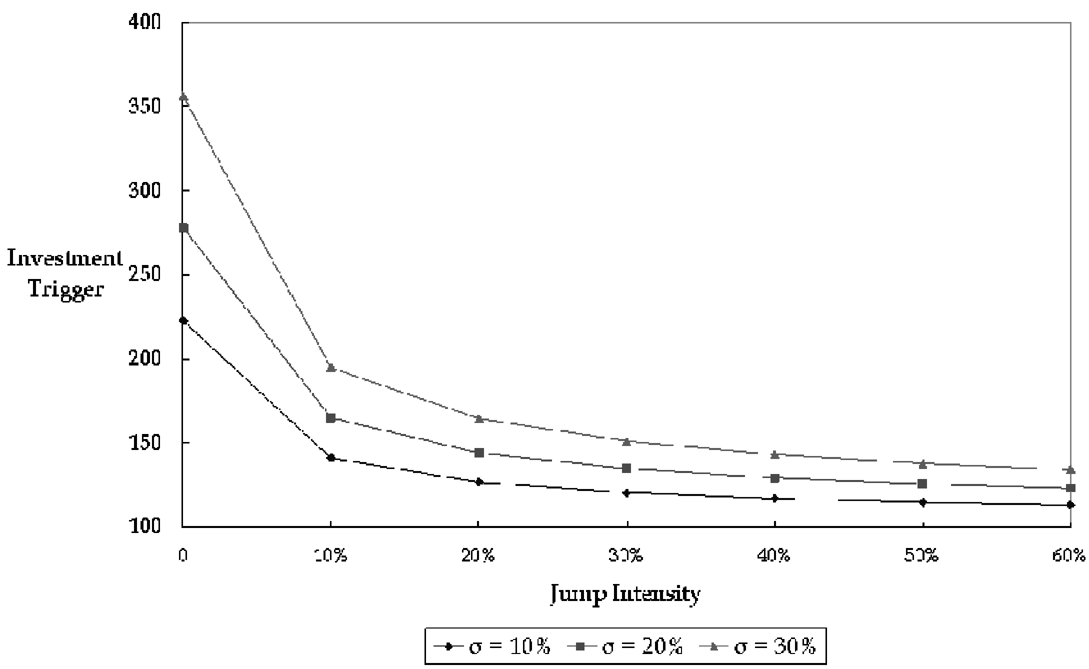
To sum up, the variance effect has a negative impact on investment due to the higher optimal investment triggers, while the realization effect can have a positive or negative impact on investment, depending on the combinations of a variety of parameter values. Consequently, the overall effect of these two offsetting forces on investment is nonlinear. The numerical analysis indicates that uncertainty may in effect encourage investment for a lower level of volatility and discourage investment for a higher level of uncertainty. Furthermore, a greater opportunity cost may lower optimal investment trigger, leading to a positive impact on investment.

4.2 The Uncertainty-Investment Relationship under an MX Process

Another stochastic process of interest is a mixed diffusion-jump process, which is a mixture of a GBM and a Poisson down jump, proposed by McDonald and Siegel (1986) and Dixit and Pindyck (1994). Since the mixed diffusion-jump process contains an additional source of uncertainty, Poisson down jumps, it is necessary to analyze the "jump effect" on investment in addition to the variance effect. The jump effect on investment can be defined as the effect of increased jump arrivals on optimal investment triggers, other parameters being constant. As a comparison to the project under a GBM, the same parameter values in the base case are also applied in the numerical analysis. The jump effect is exhibited in Figure 6.

As shown in Figure 6, an increase in the rate of jump intensity lowers the optimal investment triggers, holding the volatility unchanged. This finding suggests that an increase in jump intensity leads to a positive effect on investment. The intuition is that management should undertake investment sooner when there is an increasing probability of jump, meaning a higher intensity of competitive arrivals. Contrast to the jump effect, the variance effect under a mixed diffusion-jump process still holds, suggesting that increased volatility has a negative impact on investment. This is possibly because only down jumps allowed are in this specific form of mixed diffusion-jump process.

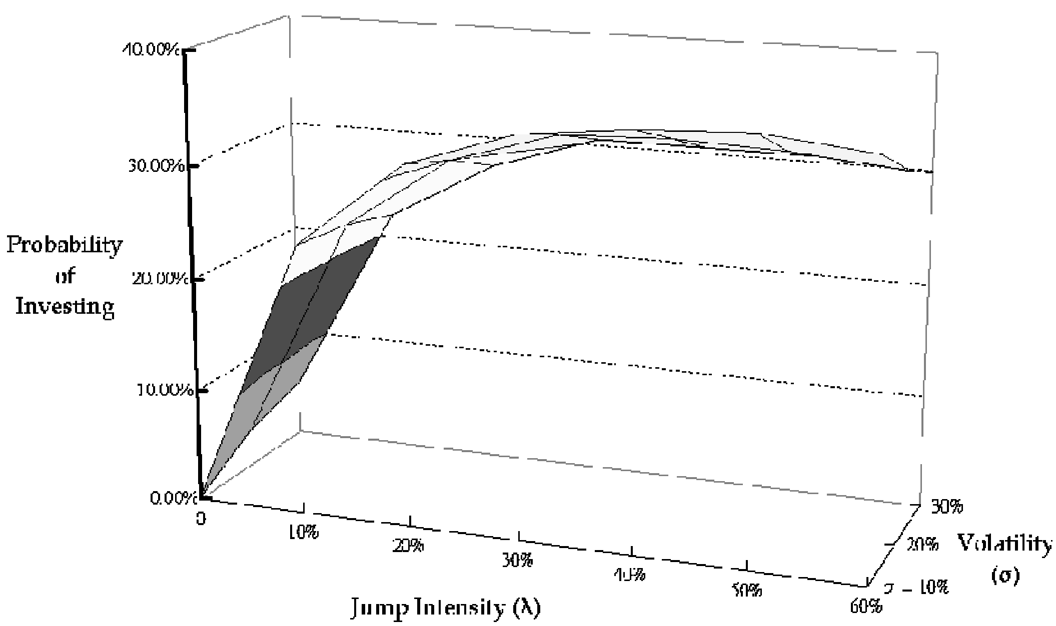
Figure 6
The Jump Effect under a Mixed Diffusion-Jump Process



To further illustrate how the combined uncertainty of both volatility and jump influence investment, Monte Carlo simulation is then conducted to evaluate the probability of investing. Figure 7 provides the result of the probability of investing as a function of jump intensity. According to Figure 7, the probability of investing appears to be a hump-shaped curve as jump intensity increases, holding the volatility constant. For a lower level of jump intensity, the probability of investing is initially an increasing function of jump intensity, but after a certain point the probability of investing becomes a slowly decreasing function of jump intensity. For example, given $\sigma = 20\%$, the probability of investing appears to increase for a smaller jump intensity, e.g., λ<30% and to decrease for a larger jump intensity, e.g., λ>30%. Consequently, the overall effect of three forces on investment under a MX process appears to be an inverted U-shaped function.

Figure 7 also reveals another interesting fact that the probability of investing under an MX process is significantly higher than that under a GBM process. The intuition behind the result is that as long as the probability of competitive entry is greater than zero, it is disadvantageous to defer the project infinitely and thus management is forced to launch the project sooner in order to preempt potential competitors.

Figure 7
The Probability of Investing as a Function of Jump Intensity (λ) Given an MX Process



To sum up, there are three major findings in the numerical analysis. First, the jump effect may result in a lower optimal investment trigger, thus suggesting that the jump uncertainty may encourage investment. This result is contrary to the variance effect, which has a negative effect on investment. Second, the overall effects of combining the variance effect, the jump effect, and the realization effect, on investment appear to be an inverted U-shaped function, similar to the GBM case. Consequently, increased jump uncertainty under an MX process could encourage investment in a way similar to increased volatility uncertainty. Third, it is also demonstrated that the probability of investing appears to be larger than that in the GBM case, with the competitive entry as a down jump taken into account. Therefore, increased uncertainty in terms of additional down jumps could have a positive impact on investment, contrary to conventional wisdom.

4.3 The Uncertainty-Investment Relationship under an MR Process

Metcalf and Hassett (1995) and Sarkar (2003) investigate the relationship between uncertainty and investment under a mean-reverting process. Metcalf and Hassett (1995) argue that mean reversion has two opposing effects, the variance effect and the realized price effect, on investment, and the overall effect of these two forces are approximately equal to such an extent that mean reversion can be justified by the common assumption of a GBM process. Sarkar (2003) extends their analytical framework to incorporate another effect of mean reversion, termed the risk-discounting effect of systematic risk, and thus contends that mean reversion in fact has a major (either positive or negative) impact on investment, depending on the combination of set of parametrical values, such as project duration, cost of investing, and interest rate.

Since a mean-reverting process complicates the uncertainty-investment analysis by the additional effect of mean reversion, the following analysis further extends Sarkar's study to examine how the force of mean reversion impacts on investment. It is therefore sensible to evaluate the "mean-reverting effect", in addition to the variance effect and the realization effect under a GBM process. The mean-reverting effect is defined as the influence of increased speed of mean reversion on optimal investment trigger, holding other parameters constant.

Consider the same investment project in the preceding base case. The optimal investment triggers under an MR process are derived according to Equations (12) and (13). Figure 8 displays the sensitivity of the optimal investment triggers, V^*_{MR}, to the changes in volatility and speed of mean reversion. As revealed from the diagram, V^*_{MR} appears to be a decreasing function of mean-reverting speed, implying that an increase in the speed of mean reversion leads to a decrease in the optimal investment trigger. This inverse relationship between speed of mean reversion and optimal investment trigger suggests that mean reversion results in a lower investment trigger, which increases the probability of project value exceeding investment trigger.

Why a faster speed of mean reversion tends to lower optimal investment trigger? As stated in Metcalf and Hassett (1995), increased speed of mean reversion may lead to a decreases in the long-run volatility of project value and in effect lower optimal investment trigger. It is therefore important to distinguish instantaneous volatility (or conditional volatility) from long-run volatility (or unconditional volatility) in the mean-reverting case. Consequently, even though a project under an MR process has the same instantaneous volatility as that under a GBM, the project under an MR process tends to have a smaller long-run volatility due to the property of mean reversion.

On the other hand, the mean-reverting effect on lowering the optimal investment trigger at a lower level of mean reversion speed is stronger than that at a higher level of mean reversion speed. This result is mainly because the optimal investment trigger at a higher level of mean-reverting speed implies a lower unconditional volatility, thus reducing the option value.

As shown in Figure 8, another characteristic regarding mean reversion is that increased instantaneous volatility can raise optimal investment trigger in a similar way under a GBM, holding the speed of mean reversion and other parameters constant. Therefore, the variance effect has the same negative impact on investment by raising the optimal trigger, a finding independent of the assumption of stochastic process.

As we further examine the realization effect under an MR process by conducting Monte Carlo simulation, the probability of investing as a function of volatility and mean-reverting speed is exhibited in Figure 9. There are two major findings that can be drawn from the diagram. First, the probability of investing under a mean-reverting process appears to be an increasing function of volatility for the base case, holding the speed of mean reversion constant. Consequently, the inverted U-shaped relationship between uncertainty and investment is less significant in the MR case. Second, similar to the other stochastic processes, the realization effect of volatility could have positive influence on investment, suggesting that increased (instantaneous) volatility under an MR process could encourage investment due to a higher probability of investing.

Figure 8
The Mean-Reverting Effect under a Mean-Reverting Process

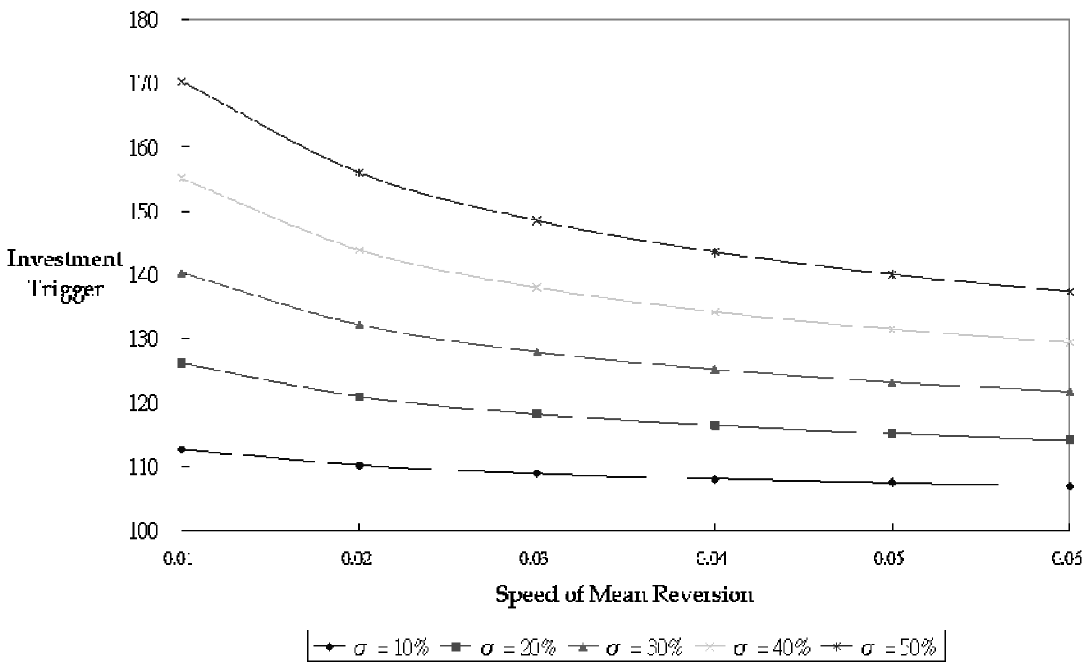
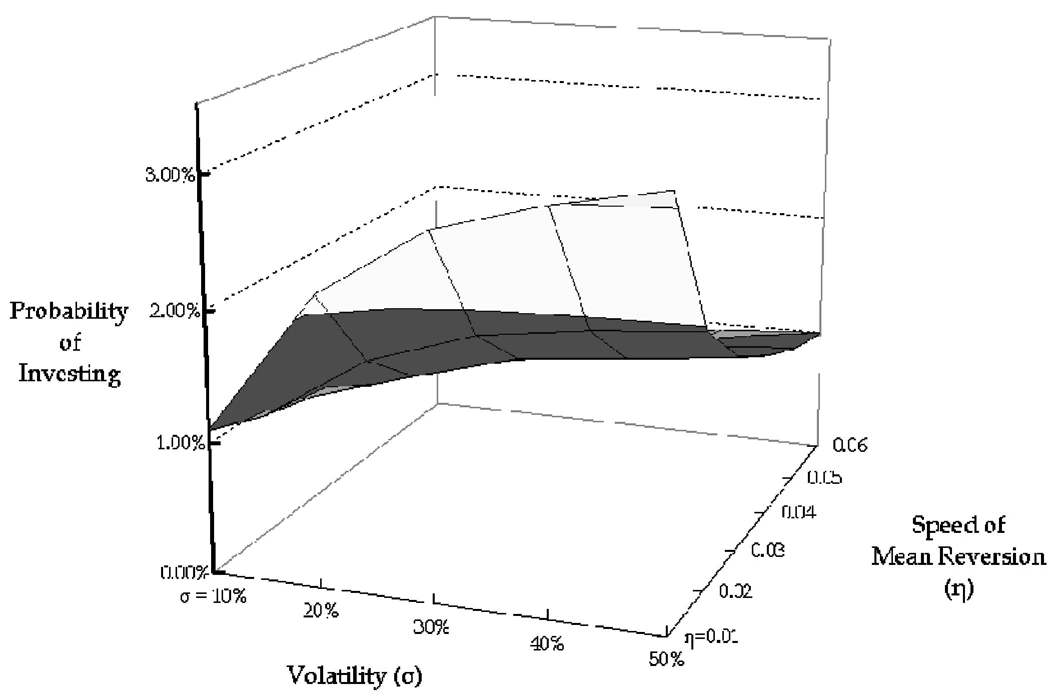
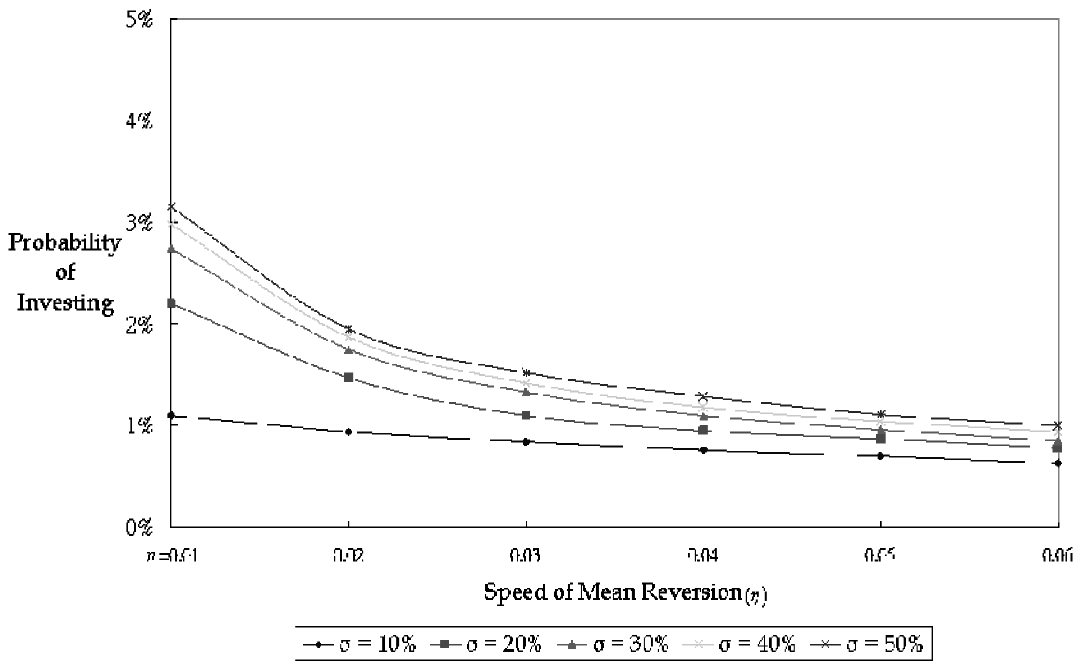


Figure 9
The Probability of Investing under a Mean-Reverting Process as a Function of Volatility (σ) and Speed of Mean Reversion (η)



It is also interesting to examine how mean reversion influences the realization effect on investment. Figure 10 displays the probability of investing as a function of mean-reverting speed. As illustrated in Figure 10, the probability of investing appears to be a convex, decreasing function of mean-reverting speed for all the levels of instantaneous volatility, suggesting that the realization effect under an MR process may reduce the chance to invest as the speed of mean reversion increases. The evidence drawn from Figure 10 is consistent with the finding in Sarkar (2003, p.388).[9] However, this result, contrast to Sarkar (2003), still holds regardless of the presence of the "risk-discounting effect".

Figure 10
The Probability of Investing under a Mean-Reverting Process as a Function of Mean-Reverting Speed (η)



To sum up, increased uncertainty under an MR process has three major effects on investment, namely, the mean-reverting effect, the variance effect, and the realization effect. The mean-reverting effect has an inverse impact on optimal investment trigger, hence leading to an increasing probability to invest. The variance effect incurs an increase in optimal investment trigger, thus implying a negative impact on investment. Combining the realization effect to the former effects, increased uncertainty under a mean-reverting process is found to have a positive impact on investment. However, although mean reversion may lower the long-run volatility and thus reduces optimal investment trigger, yet increased mean-reverting speed also diminishes the likelihood of reaching the optimal trigger such that the overall effect under an MR process may have a negative impact on investment.

4.4 The Uncertainty-Investment Relationship under a Jump Amplitude Process

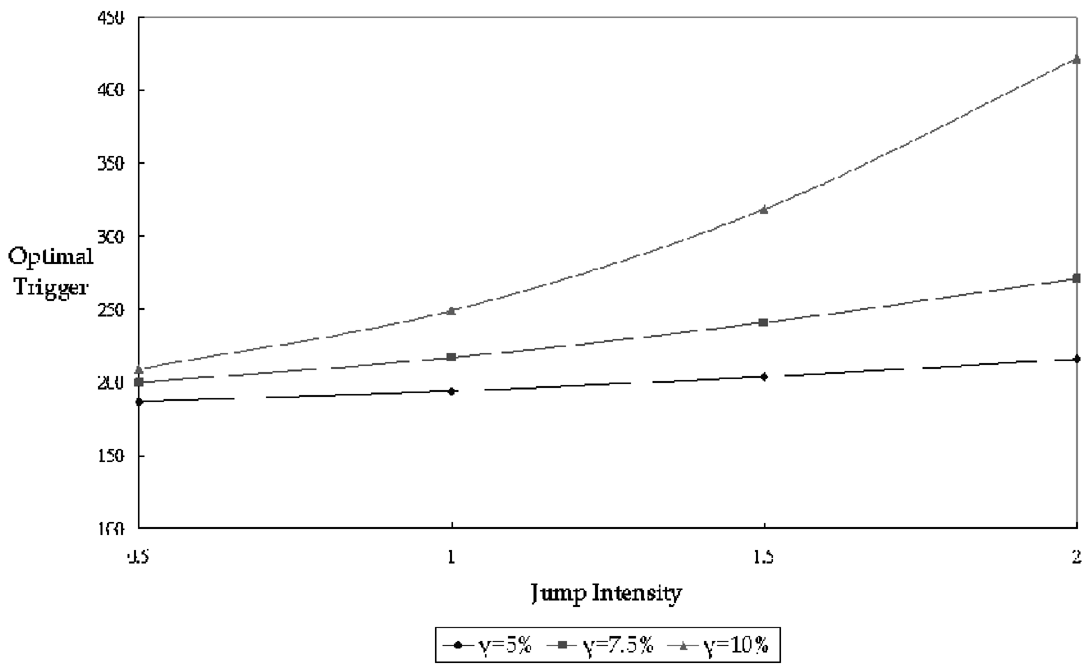
A jump amplitude process is characterized by stochastic jumps in a setup that both of the jump direction and the jump size are simply random. Since there is no closed-form solution for optimal investment trigger under a JA process, the technique of Monte Carlo simulation proposed in the preceding section could also be applied to derive V^*_{JA}. The main source of uncertainty is stochastic

[9] Sarkar (2003, p.388) states that mean reversion tends to have a positive (negative) impact on investment for long-lived (short-lived) projects, holding others constant. The short-lived project in his numerical analysis is 5 years, same as the base case our study here.

jumps under a jump-amplitude process, so numerical analysis is directed at examining the overall effect of two opposing forces, i.e., the jump effect and the realization effect. The jump effect under a JA process describes how stochastic jumps impact on V^*_{JA} and the realization effect measures the probability of V_T exceeds V^*_{JA}.

With the parametrical values in the base case, the jump effect on optimal investment trigger, by varying the jump intensity (λ) and the mean jump size (γ), is presented in Figure 11. In Figure 11, the increase in the jump intensity is shown to raise the optimal investment trigger, thus suggesting a negative impact on investment. Furthermore, an increase in the jump size leads to an increase in V^*_{JA}, holding the jump intensity and the other parameters constant. As both jump size and jump intensity increase, the jump effect on raising V^*_{JA} becomes more obvious due to an increase in option value, hence leading to a convex, increasing function of both of the jump intensity and the jump size.

Figure 11
The Jump Effect under a Jump Amplitude Process



To further examine the realization effect, Monte Carlo simulation is conducted to evaluate the probability of investing. Figure 12 presents the sensitivity of the probability of investing to the changes in jump intensity for three different levels of jump size. As exhibited in Figure 12, the probability of investing appears to be an inverted U-shaped curve as the jump intensity increases. The probability of investing indicates an increasing function of jump intensity at a lower level of jump intensity, but after a certain point of jump intensity, the probability of investing becomes a decreasing function of jump intensity. The uncertainty-investment relationship under a JA process can be expressed as follows:

$$\frac{\partial P(Inv)\big|_{JA}}{\partial \lambda} > 0 \text{, for } \lambda < \lambda^*, \quad (34)$$

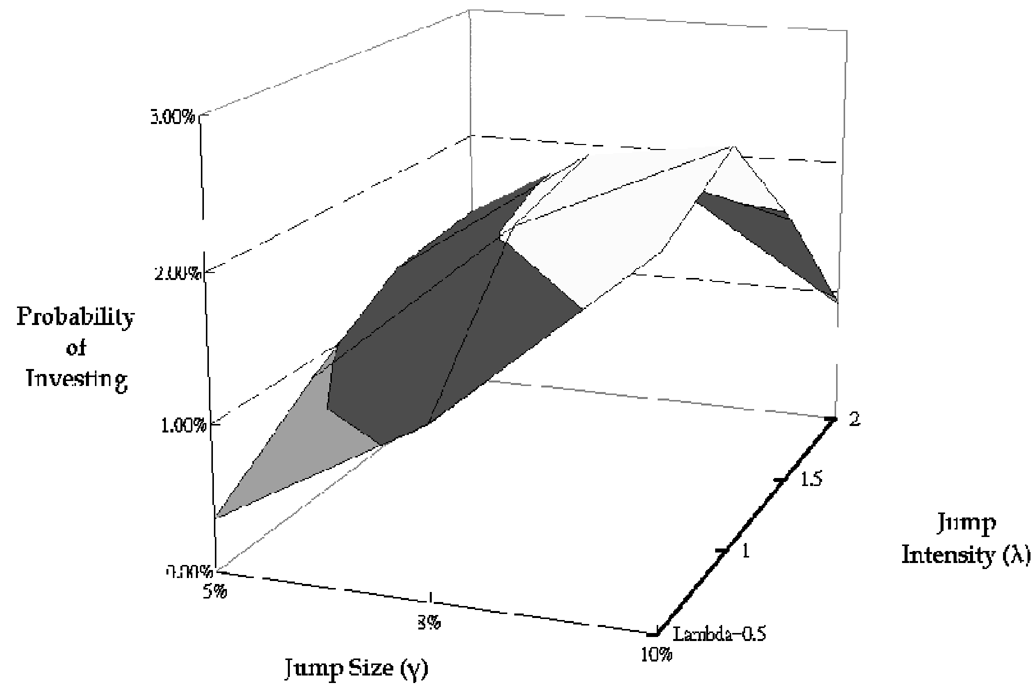
and

$$\frac{\partial P(Inv)\big|_{JA}}{\partial \lambda} < 0 \text{, for } \lambda > \lambda^{*}. \qquad (35)$$

where $V_{t+\Delta t} = V_t\left[e^{(v\Delta t+\sigma\sqrt{\Delta t}\varepsilon_t)}\right]$ denotes the point of jump intensity which peaks the probability of investing.

Figure 12 also reveals that the curve of the probability of investing climbs up as the jump size increases. Since the jump size indicates another form of uncertainty, this means that increased uncertainty may increase the probability of investing. As a result, the overall effect of uncertainty under a JA process does not necessarily discourage investment.

Figure 12
The Probability of Investing under a Jump Amplitude Process as a Function of Jump Intensity (λ) and Jump Size (γ)



5. Concluding Remarks

Conventional belief in a negative relationship between uncertainty and investment has dominated investment theory for a long time. This paper postulates an argument that increased uncertainty, in certain situations, may actually encourage investment. Since earlier studies mostly base their arguments on the GBM assumption, the study extends the assumption to alternative stochastic processes, e.g., MX, MR, and JA processes, and finds that increased uncertainty in terms of different sources may encourage investment. The overall effect of uncertainty on investment is interpreted by the probability of investing, and found to be an inverted U-shaped relationship between uncertainty and investment. Our finding is consistent with the conclusion in Lensink and Murinde (2006) with an extension of more underlying stochastic processed considered.

The study proposes the technique of Monte Carlo simulation to derive the optimal investment trigger and the probability of investing. The overall effect of uncertainty on investment is analyzed by decomposing the overall effect into the variance effect and the realization effect. The former describes the effect that increased uncertainty raises the optimal investment trigger, thus discouraging investment; while the latter states that increased uncertainty may in effect increase the probability of investing, thus encouraging investment. For the other stochastic processes, additional source of uncertainty is also explored as it may complicate the overall effect on investment. There are several additional effects under alternative processes. First, it is demonstrated that the jump effect under an MX process may lower optimal investment trigger, thus leading to a positive impact on investment. Second, the effect of mean reversion under an MR process may lower the optimal investment trigger, thus leading to a positive impact on investment. Third, the effect of stochastic jumps under a JA process are complicated by its jump intensity and jump size, both of which raise the optimal investment trigger, thus resulting in an inverse impact on investment.

The managerial implication of the study is that uncertainty does not always discourage investment even in the presence of several risk sources such as random walk and stochastic jumps. Furthermore, it is obvious that the high-risk projects are not always dominated by the low-risk projects because the high-risk projects may have a positive realization effect due to a higher probability of exceeding the optimal investment trigger, leading to a positive impact on investment. Management may increase firm value by choosing a high risk project due to a higher chance of success. For future study, it is suggested that the research attention could be directed to analyze how increased uncertainty impacts on aggregate investment since this study considers only an investment project at the individual firm level.

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