# Portfolio Optimization with GARCH–EVT–Copula-CVaR Models

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In portfolio optimization, conditional value-at-risk (CVaR) is a wildly adopted risk measure. However the sensitivity of CVaR constraint to tail thickness has also motivated the development of extending the basic CVaR structure to overcome the problem. In this paper, we investigate whether such extension adds value to portfolio performance. Specifically, we compare the out-of-sample portfolio returns for three portfolio optimization models: the CVaR model, the minimized GARCH–EVT–Gaussian Copula-CVaR model, and the minimized GARCH–EVT–Student's t Copula-CVaR model. The influences from different rebalancing frequencies and market conditions are also examined. The empirical results suggest that the portfolio returns from the two minimized GARCH–EVT–Copula-CVaR models outperform the returns from the CVaR model under daily and weekly rebalancing frequencies. As the rebalancing interval is extended, however, the portfolio returns from the two minimized GARCH–EVT–Copula-CVaR models decrease, demonstrating the declining benefits of adopting GARCH-EVT-Copula-CVaR framework. The robustness checks indicate that the results are statistically significant during the post-crisis period. This study provides portfolio managers some empirical observations on when sophisticated tail and dependence models may enhance portfolio returns.

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## 1. Introduction

How to allocate funds among various investable alternatives is a key question in portfolio management. To search for the optimal risk-return trade-off, Markowitz (1952) proposed a Mean-Variance (MV) framework. In the MV framework, the risk is evaluated by both the variance of an individual asset and the correlations between assets within the portfolio. The expected return, on the other side, is captured by the average of asset returns. As the MV model provides an intuitive approach to solving the asset allocation problem, it has become the foundation of modern portfolio theory.

Approaches to portfolio optimization have evolved over the past several decades. While the expected return remains a common measure of the portfolio return, some alternative risk measures have been developed in light of the structural issue associated with variance. Variance, by definition, treats upside and downside variations equivalently. Empirically, investors exhibit greater concern for downside risks than for upside windfalls. Therefore, variance cannot effectively calibrate the investors' perceptions of risk, thereby leading to potential misinterpretation. To focus on investors' asymmetric attention, many alternative risk measures such as lower semi-variance, lower semi-absolute deviation, Value-at Risk (VaR), and Conditional Value-at-Risk (CVaR) have been developed. As discussed by Ortobelli et al. (2005), there are two classes of risk measures: safety-risk measures and dispersion measures. Among those risk measures, the most popular measure is CVaR, introduced by Rockafellar and Uryasev (2000). In simplified terms, CVaR is the weighted average of the worst-case scenarios within a specified confidence interval during a given time period. It is applicable to non-normal, asymmetric data and emphasizes downside risk. The advantage of using CVaR for risk diagnosis is that it is a continuous, convex, and coherent measure. CVaR also takes both the size and the probability of the loss into consideration, which represents an advantage over VaR (Chen, Fabozzi, and Huang, 2012). Owing to its attractive mathematical properties, CVaR or CVaR-based modeling has been one of the most popular risk measures used for portfolio optimization (Kolm, Tütüncü, and Fabozzi, 2014).

However, as CVaR is the conditional expectation of the loss above VaR, the accuracy of the CVaR measure is highly dependent on the tail modeling and data probability distribution. This considerable need for an improved tail calibration and dependence measure led to follow-up studies such as Wang, Chen, Jin, and Zhou (2010) and Deng, Ma, and Yang (2011), which incorporate sophisticated tail modeling when applying the CVaR risk measure.

In this paper, we adopt a different perspective to examine the imposition of a CVaR constraint in portfolio optimization. According to DeMiguel, Garlappi, and Uppal (2009), sophisticated models do not consistently deliver better performance than the naïve model in portfolio optimization; therefore, we wish to evaluate whether extending the CVaR risk measure with a sophisticated tail treatment and advanced dependence modeling creates a model that outperforms the plain CVaR risk measure. In other words, are additional mathematical treatments worth the effort? Should portfolio managers consider using extended CVaR models to achieve better portfolio performance? If not, what are the reasons for the inefficiency of the sophisticated computations? Do states of economy and portfolio rebalancing frequencies affect the potential benefit of adapting advanced mathematical treatments in such modeling? To analyze this question, we adopt a combination of a GARCH innovation process, extreme value theory (EVT), and copula dependence to develop two CVaR-based risk measures for portfolios: the GARCH–EVT–Gaussian Copula-CVaR model and the GARCH–EVT–Student's t Copula-CVaR model. The GARCH-EVT is included to calibrate the tail behaviors, and the copula is adopted to estimate the dependence between assets. Because equity dependence varies over time (Login and Solnik (2001); Ang and Chen (2002)), we adopt a rolling window technique to derive a series of dependence coefficients to reflect this empirical phenomenon. Finally, copula simulation is performed to estimate portfolio loss distributions and to calculate optimal asset allocation weights by minimizing portfolio CVaRs. To evaluate the potential contribution of advanced mathematical treatments, the out-of-sample performance of the CVaR model, the GARCH–EVT–Gaussian Copula-CVaR model, and the GARCH–EVT–Student's t Copula-CVaR model is compared. We also assess the significance of portfolio return differences using Ledoit and Wolf's (2008) non-linear studentized time series bootstrap method.

This paper contributes to the existing studies in the following ways. First, it offers a comprehensive examination of the use of the CVaR constraint in portfolio optimization. Despite the well-acknowledged benefits of using GARCH-EVT, copula, or CVaR models in empirical finance, few studies have managed to combine the three structures, i.e., GARCH-EVT to calibrate tail behaviors, copulas to gauge the dependence structure, and CVaR to estimate the downside risk, into a composite framework due to the complex programming involved. Moreover, many of the existing studies are based on a static framework, leaving the time–varying equity dependence out of the consideration. In this study, we not only illustrate how to integrate the three structures but also refine previous analysis by adding a rolling window method to capture the time-varying property of equity dependence. Second, this study offers portfolio managers a reference for whether quantitative models may provide better portfolio performance. We compare the performance of three portfolio optimization models under four different rebalancing intervals for both recession and expansion periods. Our empirical results indicate that for active or shorter rebalancing, it is necessary to apply treatments to fat-tailed data to achieve higher returns. For longer rebalancing intervals, this study does not observe consistent statistical significance in the difference between using and not using tail treatments and advanced dependence measures. The remainder of this paper is organized as follows. Section 2 reviews existing studies on portfolio optimization using the CVaR constraint. Section 3 illustrates the construction of the empirical models, and Section 4 describes the data. The main empirical results are reported and discussed in Section 5, and Section 6 concludes.

## 2. Literature Review

The M-V model developed by Markowitz (1952) offers a starting point for identifying an optimal asset allocation based on the trade-off between risk and return. Kolm, Tütüncü, and Fabozzi (2014) argued that this model represented a substantial breakthrough at the time because it posited a quantitative approach to resolving complex financial decision making. However, this intuitive approach comes at the expense of being highly sensitive to the changes in inputs, while in practice the ex-ante parameters are unknown (Levy and Levy, 2014). In recognizing such potential estimation errors, many extensions of the M-V model have been developed in an attempt to mitigate the problem, and defining the proper instrument for measuring risk and identifying the proper dependence measures has been prioritized in this regard.

In recent years, portfolio optimization using lower tail risk has become popular as investors express greater concern regarding downside risk. According to Artzner et al. (1999), a proper risk measure should be coherent, i.e. satisfy four properties: translation invariance, subadditivity, positive homogeneity, and monotonicity. Although VaR is a wildly used risk measure in various industries, it is not a coherent measure and is subject to several mathematical disadvantages. CVaR, by contrast, is a modification of VaR that eliminates the latter's undesirable properties. CVaR is defined as the weighted average of the worst-case scenarios within a specified confidence interval during a given time period. Since Pflug (2000) proved that CVaR is a coherent measure, CVaR has become a popular tool in estimating risk, and studies have begun to use CVaR in portfolio optimization.

Rockafellar and Uryasev (2000) introduced linear programming and nonsmooth optimization techniques in portfolio optimization with CVaR constraints. They illustrated that the proposed techniques can efficiently reduce CVaR and are suitable for application in risk management. Alexander and Baptista (2004) compared VaR and CVaR constraints in portfolio optimization, concluding that a CVaR constraint is tighter and more efficient than a VaR constraint in most situations and that the CVaR constraint tends to be a better risk management tool than a VaR constraint. Alexander, Coleman, and Li (2006) studied CVaR minimization problems when constructing a derivatives portfolio. They found that by introducing cost as an additional parameter in the CVaR optimization, an optimal CVaR portfolio will have fewer instruments, leading to lower transaction costs. Chen, Fabozzi, and Huang (2012) illustrated how transaction costs affect the portfolio revision when mean-CVaR is used as the risk measure. They demonstrated how to integrate CVaR to deal with the computational difficulty involved in identifying a robust portfolio. Tong, Qi, Wu, and Zhou (2010) utilized a smoothing technique when solving portfolio optimization using a CVaR constraint, concluding that the smoothing method is appropriate for any portfolio optimization involving semi-smooth cases.

Copula functions are known for their applicability to a non-elliptical data distribution. As such, incorporating copula functions into the CVaR optimization process is also increasingly common in the literature. Yu, Yang, and Li (2009) introduced a variance Gamma (VG) copula approach and applied it in a minimized CVaR portfolio model. Using three Chinese stock indices in a portfolio, their study demonstrated that the conventional Gaussian copula is not able to capture the skewness and kurtosis of assets returns and showed that the VG process may be a better alternative. Wang, Chen, Jin, and Zhou (2010) applied the GARCH-EVT-Copula models in estimating the VaR and CVaR of foreign exchange portfolios, finding that the Student's t and Clayton copulas are more accurate in measuring dependence than the Gaussian copula is. Deng, Ma, and Yang (2011) adopted the pair Copula-GARCH-EVT-CVaR models in optimizing portfolios of four Chinese stock indices, concluding that employing the pair copula in lieu of the Student's t copula leads to better portfolio performance. Kakouris and Rustem (2014) demonstrated how to combine rival copulas and CVaR to provide solutions for robust portfolio optimization. They constructed a Worse Case CVaR (WCVaR) using rival copulas and showed that the WCVaR framework is especially useful during periods of crisis.

Despite the popularity of employing CVaR in portfolio optimization, existing studies have tended to assume a static rather than a dynamic framework to avoid computational difficulties, thereby generating the potential for estimation error. Moreover, when evaluating portfolio performance, the effects of the choice of sample period and portfolio rebalancing frequencies have seldom been investigated. These hitherto unaddressed questions motivate the design of the dynamic GARCH-EVT-Copula-CVaR models in the next section.

**3. Methodology**

This section illustrates the steps for solving the portfolio optimization problem under the CVaR constraint within the multivariate GARCH-EVT-Copula framework.

**3.1 The GARCH-EVT application**

EVT is known for its ability to capture extreme tail behaviors. Studies on EVT and its application in finance include Embrechts et al. (1997), Beirlant et al. (2004), McNeil (1998), McNeil and Frey (2000), and Dowd (2005), among others. Within the EVT framework, an important assumption is that the data are required to be independent and identically distributed (iid) random variables, which is not commonly seen in return data. To fulfill this requirement, McNeil (1998) recommends using the GARCH (1, 1) model as an intermediate step to transform the original data into iid data before incorporating EVT. The GARCH model is specified as follows:

$$r_{i,t} = \mu_i + \varepsilon_{i,t} \tag{1}$$

$$\varepsilon_{i,t} = z_{i,t}\sigma_{i,t} \tag{2}$$

$$z_{i,t} \sim iid \tag{3}$$

, and

$$\sigma_{i,t}^2 = \omega + \eta_i \varepsilon_{i,t-1}^2 + \tau_i \sigma_{t-1}^2 \tag{4}$$

, where $\omega > 0$, $1 \geq \eta \geq 0$, $1 \geq \tau \geq 0$, and $\eta + \tau < 1$ are necessary conditions, and $r_{i,t}$ is the actual return from the sample. Conditioning on the information on date $t-1$, $\mu_i$ represents the conditional expected return, and $\sigma_{i,t}$ denotes the conditional volatility of return $i$ on date $t$. $z_{i,t}$ is an iid sequence that follows the Gaussian distribution.

According to Wand and Jones (1995), the standard uncorrelated residuals $Z_i = z_{i,1}, z_{i,2}, z_{i,3}, \ldots z_{i,n}$ can be converted into the marginal cumulative distribution with the Gaussian kernel estimate expressed as follows:

$$f_h(z_i) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{z_{i,t} - Z_i}{h}\right) \tag{5}$$

, where K(.) is the Gaussian kernel, as $K(\mu) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}\mu^2}$, and $h$ is a smoothing parameter that is a positive number close to zero.

To fit fat-tailed return distributions, a semi-parametric approach, i.e., a non-parametric empirical distribution calibrating the center of the return distribution and EVT describing the tails of return distributions with the parametric generalized Pareto distribution (GPD), is adopted. Thus, the marginal distribution of $Z_i$ is defined as follows:

$$F_i(\hat{z_i}) = \begin{cases} \frac{k_l}{n}\left[\hat{\xi}_i^l \frac{v_i^l - z_i}{\beta_i^l}\right]^{\frac{-1}{\xi_i^l}}, for\ z_i < v_i^l \\ \varphi(\hat{z_i}), for\ v_i^l < z_i < v_i^r \\ 1 - \frac{k_r}{n}\left[\hat{\xi}_i^r \frac{z_i - v_i^r}{\beta_i^r}\right]^{\frac{-1}{\xi_i^r}}, for\ z_i > v_i^r \end{cases} \tag{6}$$

, where superscripts $l$ and $r$ denote the left and right tails, respectively. $\beta$ denotes the scale parameter, and $\xi$ denotes the shape parameter. When $\xi<0$, it indicates that the distribution has a finite tail, and when $\xi = 0$, it indicates that the distribution has a thin tail. For $\xi>0$, it indicates that the distribution has a fat tail. $n$ is the number of observations, and $k$ represents the amount of observations beyond threshold $v$.

Identifying an appropriate threshold $v$ is key in this approach. If the threshold is set at a low value, the GPD will include excess observations, and certain observations may not belong to the extremes, which will lead to estimation bias. However, if the threshold is set too high, there may be too few observations to model the GPD, which will lead to high variance. As determining the value of the threshold involves a trade-off between bias and variance, previous studies have recommended different threshold levels or proposed various methods for determining the optimal level (e.g., Neftci (2000), McNeil and Frey (2000) and Longin and Solnik (2001)). This paper adopts the value suggested by Neftci (2000), in which the extremes are set at the upper and lower 5% of observations.

**3.2 Copulas**

A copula is a function that ties univariate marginal distribution functions together to form a multivariate distribution function. Thus, such a joint distribution function can be separated into marginal distributions that express the characteristics of each asset and a copula function that describes the interaction between assets. Because the copula function is measured independently from the marginal distributions, copula functions are not subject to the restrictions arising from the data distribution (Cherubini et al. (2004)).

According to Sklar's theorem, if the marginal distributions $F_1, F_2, F_3, \ldots, F_n$ are continuous, a unique copula will exist for a joint distribution of the marginal distribution. That is,

$$F(z_1, z_2, \ldots, z_n) = C\big(F_1(z_1), F_2(z_2), \ldots, F_n(z_n)\big) \tag{7}$$

, where $Z = (z_1, z_2, \ldots, z_n)^T$ is a vector of $n$ random variables with marginal distributions $F_1, F_2, \ldots, Fn$. In this paper, two copula functions are adopted: the Gaussian copula, which describes the overall distribution and assumes no tail dependence, and Student's t copula, which emphasizes both the center of the distribution and the symmetric tail behaviors.

**3.2.1 Gaussian Copula**

The Gaussian copula is popular because it is straightforward in both concept and computation. According to Malevergne and Sornette (2003), the Gaussian copula is applicable to stock index return data. The multivariate Gaussian copula can be written as follows:

$$C\left(F_1(z_{1,t}), F_2(z_{2,t}), \ldots, F_n(z_{n,t})\right) = \Phi_\Omega\left(\Phi^{-1}(z_{1,t}), \Phi^{-1}(z_{2,t}), \ldots, \Phi^{-1}(z_{n,t})\right) \tag{8}$$

$$\Phi_\Omega\left(\Phi^{-1}(z_{1,t}), \Phi^{-1}(z_{2,t}), \ldots, \Phi^{-1}(z_{n,t})\right) = \frac{\frac{1}{\sqrt{2\pi^n\Omega}}\exp\left(\frac{-1}{2}Z^T\Omega^{-1}Z\right)}{\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-1}{2}z_j^2\right)} \tag{9}$$

, where $\Phi$ denotes a univariate standard normal distribution function, $\Phi_\Omega$ is the joint distribution of the multivariate standard normal distribution functions, and $\Omega$ is the correlation coefficient matrix, in which the value of each correlation coefficient is between -1 and 1. Because the Gaussian copula does not focus on the tails of the distribution, tail dependence only exists when the correlation coefficient is 1.

If we define $\rho = \left(\Phi^{-1}(z_{1,t}), \Phi^{-1}(z_{2,t}), \ldots, \Phi^{-1}(z_{n,t})\right)$, $\Omega$ can be estimated via the maximum likelihood estimation (MLE) method as $\hat{\Omega} = \frac{1}{T}\sum_{t=1}^{T}\rho_t\rho_t'$.

**3.2.2 Student's t Copula**

Student's t copula is built on the $t$ distribution. The multivariate Student's t copula calibrates dependence for both the center and the tails of the distribution and is defined as follows:

$$C_{v,\Omega}^{t}\left(F_1(z_{1,t}), F_2(z_{2,t}), \ldots, F_n(z_{n,t})\right) = t_{v,\Omega}\left(t_v^{-1}(z_{1,t}), t_v^{-1}(z_{2,t}), \ldots, t_v^{-1}(z_{n,t})\right) \tag{10}$$

$$t_{v,\Omega}\left(t_v^{-1}(z_{1,t}), t_v^{-1}(z_{2,t}), \ldots, t_v^{-1}(z_{n,t})\right) = \int_{-\infty}^{t_v^{-1}(z_1)}\int_{-\infty}^{t_v^{-1}(z_2)}\cdots\int_{-\infty}^{t_v^{-1}(z_n)}\frac{\Gamma\left(\frac{\upsilon+n}{2}\right)\frac{1}{\sqrt{|\Omega|}}}{\Gamma\left(\frac{\upsilon}{2}\right)(\upsilon\pi)^{\frac{n}{2}}}\left(1+\frac{1}{v}z^T\Omega^{-1}z\right)^{-\frac{v+n}{2}}dz_1dz_2\ldots dz_n \tag{11}$$

, where $t_{v,\Omega}$ denotes the multivariate joint t distribution, $t_v^{-1}$ stands for the inverse of the distribution of a univariate $t$ distribution, $\Omega$ is the correlation coefficient matrix, and $v$ indicates the degrees of freedom. The correlation coefficient $\rho$ exists when the degrees of freedom are greater than 2. Defining $\rho = \left(t_v^{-1}(z_{1,t}), t_v^{-1}(z_{2,t}), \ldots, t_v^{-1}(z_{n,t})\right)$, $\Omega$ can be derived using the maximum likelihood estimation (MLE) method, and no flexibility will be lost when fixing the degrees of freedom. This approach can be written as follows:

$$\hat{\Omega} = \left(cov(Z) - \frac{2v^2}{(v-2)^2(v-4)}\hat{\rho}\hat{\rho}'\right)\frac{v-2}{2} \tag{12}$$

### 3.3 The Conditional Value-at-Risk (CVaR) model

Mathematically, CVaR can be defined as follows:

$$CVaR(r_p)_\beta = -E\{r_p \in R \mid r_p \leq -VaR_\beta^h\}, \quad (13)$$

and

$$VaR(r_p)_\beta{}^h = inf\{r_p \in R \mid P(L \leq \widehat{VaR}) = \alpha\} \quad (14)$$

, where $L$ is a sequence number, $r_{p,t}$, $r_{p,t-1}$, $r_{p,t-2}$, $r_{p,t-3}$, …$r_{p,t-h}$, representing the portfolio returns at times $t$, $t$-1, $t$-2, $t$-3,..., $t$-$h$, respectively. $a$ is a small percentage close to 0 (typically 1% or 5%); $\beta$ is a given level of confidence; and $h$ is the target horizon denoting the differential period between $t$ and $(t+h)$.

For an $n$-asset portfolio with $W=(w_1, w_2, \ldots, w_n)^T$ as the weight of each asset and $R_t=(r_{1,t}, r_{2,t}, \ldots, r_{n,t})^T$ as the return of each asset on date $t$, the expected portfolio return $r_{p,t}$ is defined as $\sum_{i=1}^{n} w_i r_{i,t}$, and the loss function of the portfolio is $f(w,r)=-W^TR$, with the density $p(r)$. Based on the methodology suggested by Rockafellar and Uryasev (2000), the minimized CVaR can be written as follows:

$$CVaR_\beta(w) = \min_{\alpha \in \mathbb{R}} F_\beta(w, \alpha),$$

and (15)

$$F_\beta(w,\alpha) = \alpha + \frac{1}{1-\beta}\int_{r \in \mathbb{R}^m} [f(w,r) - \alpha]^+ p(r)dr. \quad (16)$$

Thus, as explained in the following section, the optimal weights, $W$, will be calculated using three models: the Rockafellar and Uryasev (2000) CVaR model, the multivariate GARCH–EVT–Student's t Copula-CVaR model, and the multivariate GARCH–EVT–Gaussian Copula-CVaR model.

### 3.4 Portfolio return estimation

To evaluate the portfolio performance, this paper assumes no transaction costs and no short-selling constraints[1]. Such assumptions allow us to focus on the impact of the different models on the portfolio returns. In addition, to reflect the time-varying dependence structures, a 500-day rolling window technique is used to provide a sequence of parameters for the out-of-sample portfolio estimation. There are two steps involved in this approach. First, the data from date $t_1$ to date $t_{500}$ are taken to calculate the optimal weightings by solving a quadratic function subject to constraints. Second, the derived optimal weightings are applied to the return data on date $t_{501}$ to compute the out-of-sample portfolio returns for date $t_{501}$.

For the CVaR model, Rockafellar and Uryasev (2000) suggested using a linear programming (LP) approach to estimate a corresponding approximation as follows:

$$\widetilde{F_\beta}(w,\alpha) = \alpha + \frac{1}{q(1-\beta)}\sum_{k=1}^{q}[f(w,r_k) - \alpha]^+. \quad (17)$$

The optimal weights can then be solved by minimizing the Monte Carlo simulation returns through line search techniques or standard linear programming. This study sets $a=0.05$ and $\beta=0.95$ and uses the data for the prior 500 days to calculate the expected mean and expected variance of each MSCI index for the Monte Carlo simulation. The optimal weights for the out-of-sample return on $t_{501}$ are solved based on a line search using a Monte Carlo simulation with 10,000 simulated observations (q=10,000).

In the two GARCH–EVT–Copula-CVaR models, three steps are involved. First, the data for the prior 500 days are used to independently estimate the correlation coefficient matrices of the multivariate Gaussian copula or Student's t copula functions. Second, $\hat{\Omega}$ is decomposed, and 10,000 residuals are simulated. The residuals are then restored through the GARCH (1, 1) model to generate the return series of each MSCI index. Finally, the confidence level is set to 95% to calculate portfolio CVaRs with various weights and to determine the optimal weights for the out-of-sample return on $t_{501}$ through the minimum CVaR.

[1] Short-selling typically entails additional costs for investors, and studying the effect of investment constraints is a different line of research. To focus on the influence of portfolio modeling on portfolio performance, this paper adopts a no short-selling constraint to simplify the comparison.

### 3.5 Portfolio Rebalancing

To maintain an efficient asset allocation across horizons, periodic portfolio rebalancing is important because it helps the portfolio re-adjust to its original goal and, therefore, reduce investors' exposure to unnecessary risks. The crucial question for portfolio managers is how often a portfolio should be rebalanced and whether the choice of the optimization model affects the optimal portfolio rebalancing frequency. According to Eaker and Grant (2002), the choice of portfolio rebalancing strategy has a substantial impact on portfolio performance. They documented that the benefits of portfolio rebalancing decrease as the rebalancing frequency increases. Mendes and Marques (2012) examined twelve rebalancing strategies, varying in the rebalancing frequencies and in target risk tolerance, and concluded that semi-annual rebalancing provides the best long return performance. In this study, we follow the assumption in Mendes and Marques (2012) of not considering rebalancing costs and empirically examine the dynamics of four portfolio rebalancing strategies--daily, weekly, bi-weekly, and monthly--and their impacts on the selection of portfolio optimization models.

## 4. Data

The data adopted in this study are the U.S. dollar-denominated daily MSCI indices for the G8 countries (Canada, France, Germany, Italy, Japan, Russia, the U.K., and the U.S.). The sample period ranges from the first trading day in December 2005 to the last trading day in January 2012 and consists of a total of 1,587 observations. With a 500-day rolling window, the out-of-sample period covers 1,087 returns, ranging from the first trading day in December 2007 to the last trading day in January 2012. To examine the model's performance across recession and expansion periods, the sample period is categorized into two sub-periods. Following the announcements made by the National Bureau of Economic Research (NBER), this paper defines the first period as the financial crisis period, ranging from December 3, 2007 to June 30, 2009, and the second period as the post-crisis period, ranging from July 1, 2009, to January 31, 2012.

The daily returns, $r_{i,t}$, of the MSCI index for each G8 country are calculated as follows:

$$R_{i,t} = LN(\frac{I_{i,t}}{I_{i,t-1,}}) \tag{18}$$

, where $I$ =1 to 8, representing each country in the G8.

The descriptive statistics for the G8 MSCI indices are reported in Table 1. The average returns for the G8 countries are all negative. Thus, although the markets rebounded in the post-crisis period, the overall portfolio gains for the entire period are not sufficiently high to cover the losses incurred during the financial crisis period. The results regarding skewness and kurtosis indicate that the returns for the G8 countries have non-normal distributions. Japan, Russia, the U.K., and the U.S. have long distribution tails on the left-hand side, and Canada, France, Germany, and Italy have long distribution tails on the right-hand side. The results of the Jarque-Bera test also confirm that the returns for the G8 countries are not normally distributed at the 1% significance level. Therefore, a GARCH-EVT model may better fit the data.

**Table 1: Summary Statistics for the G8 MSCI Indices**

| | Canada | France | Germany | Italy | Japan | Russia | U.K. | U.S. |
|---|---|---|---|---|---|---|---|---|
| Mean (%) | -0.0098 | -0.0543 | -0.0452 | -0.0894 | -0.0343 | -0.0504 | -0.0361 | -0.0097 |
| Std. Dev. | 0.0201 | 0.0225 | 0.0219 | 0.0239 | 0.0171 | 0.0319 | 0.0201 | 0.0175 |
| Skewness | 0.6674 | 0.0938 | 0.0517 | 0.0366 | -0.1261 | -0.3754 | -0.0196 | -0.2487 |
| Kurtosis | 8.970 | 6.982 | 6.559 | 6.249 | 8.131 | 14.890 | 8.583 | 9.279 |
| Jarque-Bera | 1693*** | 719.2*** | 573.7*** | 478*** | 1194*** | 6423*** | 1410*** | 1794*** |

Note: This table reports summary statistics for the daily return series from the G8 MSCI indices. There are a total of 1,087 returns for each country. The significance of the Jarque-Bera (JB) test results is marked with asterisks, where ***, **, and * denote significance at the 1%, 5% and 10% levels, respectively. These results indicate that the daily returns of the G8 MSCI indices are not normally distributed.

## 5. Empirical Results

### 5.1 Average Portfolio Returns

Tables 2 to 5 present the average out-of-sample returns and their corresponding standard deviations and Sharpe ratios for the entire sample period, the financial crisis period, and the post-crisis period at different rebalancing frequencies.

**Table 2: Average Portfolio Returns and Sharpe Ratios under a Daily Rebalancing Strategy**

| | CVaR | GARCH–EVT–Gaussian Copula-CVaR | GARCH–EVT–Student's t Copula- CVaR |
|---|---|---|---|
| **Panel A: Average returns** | | | |
| Entire sample period | -26.9185% | 7.6631% | 13.1479% |
| | (0.5422) | (0.3229) | (0.3495) |
| The financial crisis period | -64.3859% | -22.4191% | -15.6989% |
| | (0.5834) | (0.1769) | (0.1899) |
| The post-crisis period | -4.0155% | 26.0516% | 30.7813% |
| | (0.3574) | (0.2444) | (0.3048) |
| **Panel B: Sharpe ratios** | | | |
| Entire sample period | -0.4963 | 0.2372 | 0.3761 |
| The financial crisis period | -1.1035 | -1.2668 | -0.8262 |
| The post-crisis period | -0.1121 | 1.0658 | 1.0098 |

Notes: The average daily out-of-sample returns and the corresponding standard deviations (in parentheses) are reported in Panel A. Panel B reports the corresponding Sharpe ratios. All numbers are presented in annualized formats.

**Table 3: Average Portfolio Returns and Sharpe Ratios under a Weekly Rebalancing Strategy**

| | CVaR | GARCH–EVT–Gaussian Copula CVaR | GARCH–EVT–Student's t Copula CVaR |
|---|---|---|---|
| **Panel A: Average returns** | | | |
| Entire sample period | -17.2205% | -6.8317% | -5.4367% |
| | (0.4222) | (0.2930) | (0.2874) |
| The financial crisis period | -50.0327% | -33.7907% | -31.0144% |
| | (0.4253) | (0.2543) | (0.2455) |
| The post-crisis period | 2.8369% | 9.6477% | 10.1983% |
| | (0.2658) | (0.1651) | (0.1784) |
| **Panel B: Sharpe ratios** | | | |
| Entire sample period | -0.4078 | -0.2331 | -0.1889 |
| The financial crisis period | -1.1763 | -1.3287 | -1.2631 |
| The post-crisis period | 0.1064 | 0.5838 | 0.5711 |

Notes: The average weekly out-of-sample returns and the corresponding standard deviations (in parentheses) are reported in Panel A. Panel B reports the corresponding Sharpe ratios. All numbers are presented in annualized formats.

Under a daily rebalancing strategy, the portfolios using the GARCH-EVT-Gaussian Copula-CVaR and GARCH-EVT-Student's t Copula-CVaR models outperform those using the CVaR model with respect to returns. During the financial crisis period, all three models yield negative average returns, with the GARCH-EVT-Student's t Copula-CVaR model yielding the highest average returns of -15.6989%. During the post-crisis period, the average returns from all of the portfolio models improve, with the GARCH-EVT-Student's t Copula-CVaR model still yielding the highest average returns of 30.7813%. In either period, the CVaR model provides the lowest portfolio returns. There are two implications from this result. First, under daily rebalancing, investors can achieve better portfolio performance by adopting the two GARCH–EVT–Copula models regardless of the market conditions. Second, coincidental with the findings in Kole, Koedijk, and Verbeek (2007) that Student's t copula, which considers the dependence both in the center and the tail of the distribution, provides the best fit for the extreme negative returns of the empirical probabilities.

**Table 4: Average Portfolio Returns and Sharpe Ratios under a Bi-Weekly Rebalancing Strategy**

| | CVaR | GARCH–EVT–Gaussian Copula-CVaR | GARCH–EVT–Student's t Copula CVaR |
|---|---|---|---|
| **Panel A: Average returns** | | | |
| Entire sample period | -11.6614% | -12.5087% | -9.6027% |
| | (0.4162) | (0.2671) | (0.2512) |
| The financial crisis period | -46.0786% | -35.0794% | -31.0900% |
| | (0.3857) | (0.2516) | (0.2372) |
| The post-crisis period | 9.3771% | 1.2881% | 3.5319% |
| | (0.2675) | (0.1616) | (0.1478) |
| **Panel B: Sharpe ratios** | | | |
| Entire sample period | -0.2801 | -0.4679 | -0.3821 |
| The financial crisis period | -1.1944 | -1.3938 | -1.3107 |
| The post-crisis period | 0.3502 | 0.0792 | 0.2388 |

Notes: The average bi-weekly out-of-sample returns and the corresponding standard deviations (in parentheses) are reported in Panel A. Panel B reports the corresponding Sharpe ratios. All numbers are presented in annualized formats.

**Table 5: Average Portfolio Returns and Sharpe Ratios under a Monthly Rebalancing Strategy**

| | CVaR | GARCH–EVT–Gaussian Copula-CVaR | GARCH–EVT–Student's t Copula- CVaR |
|---|---|---|---|
| **Panel A: Average returns** | | | |
| Entire sample period | -7.5846% | -13.2631% | -13.0835% |
| | (0.3837) | (0.2287) | (0.2573) |
| The financial crisis period | -40.0274% | -30.2851% | -32.9888% |
| | (0.3336) | (0.2248) | (0.2586) |
| The post-crisis period | 12.3977% | -2.8579% | -0.9159% |
| | (0.2529) | (0.1577) | (0.1637) |
| **Panel B: Sharpe ratios** | | | |
| Entire sample period | -0.1975 | -0.5797 | -0.5083 |
| The financial crisis period | -1.2071 | -1.3469 | -1.2753 |
| The post-crisis period | 0.4899 | -0.1807 | -0.0555 |

Notes: The average monthly out-of-sample returns and the corresponding standard deviations (in parentheses) are reported in Panel A. Panel B reports the corresponding Sharpe ratios. All numbers are presented in annualized formats.

When the rebalancing frequency is extended from daily to weekly, the two minimized GARCH–EVT-Copula-CVaR portfolios continue to outperform the portfolios using the CVaR model across the different states of the economy. However, the differences in the average portfolio returns between the two minimized GARCH–EVT-Copula-CVaR models and the CVaR model decrease as the rebalancing frequency decreases. GARCH-EVT-Student's t Copula-CVaR model, again, delivers the highest portfolio returns.

As the rebalancing frequency is extended to bi-weekly and monthly bases, the advantages of adopting the GARCH-EVT-Copula-CVaR models in the entire sample period diminish as the two

GARCH-EVT-Copula-CVaR models do not consistently yield the higher returns. Under bi-weekly and monthly rebalancing, the CVaR has the best portfolio performance during the post-crisis period. Therefore, the implications of the empirical evidence are that sophisticated models such as the two minimized GARCH–EVT–Copula-CVaR models will offer higher portfolio returns if the portfolio weightings are to be adjusted on a daily or weekly basis. If frequent weighting adjustments are not available, then the CVaR model should be adopted for modeling the portfolios.

**Table 6: Test of the Differences in the Portfolios' Sharpe Ratios**

| | GAU-CVaR | T-CVaR | T-GAU |
|---|---|---|---|
| **Panel A: Daily** | | | |
| Entire sample period | 0.0002*** | 0.0002*** | 0.0184*** |
| Financial crisis period | 0.2569 | 0.2292 | 0.065* |
| Post-crisis period | 0.0006**** | 0.0008*** | 0.2909 |
| **Panel B: Weekly** | | | |
| Entire sample period | 0.0062*** | 0.007*** | 0.2909 |
| Financial crisis period | 0.0656* | 0.4675 | 0.2356 |
| Post-crisis period | 0.0046*** | 0.0102*** | 0.831 |
| **Panel C: Biweekly** | | | |
| Entire sample period | 0.0454*** | 0.0370*** | 0.0354*** |
| Financial crisis period | 0.1038 | 0.0414*** | 0.3199 |
| Post-crisis period | 0.0098*** | 0.4561 | 0.0686* |
| **Panel D: Monthly** | | | |
| Entire sample period | 0.0026*** | 0.0008*** | 0.0314*** |
| Financial crisis period | 0.2713 | 0.4027 | 0.7383 |
| Post-crisis period | 0.0002*** | 0.0002*** | 0.0190*** |

Note: Following the framework proposed by Ledoit and Wolf (2008), this paper tests whether the portfolios' Sharpe ratios are statistically different. In the table below, T denotes the GARCH–EVT–Student's t Copula-CVaR model, GAU denotes the GARCH–EVT–Gaussian Copula-CVaR model, and CVaR denotes the original minimized CVaR model. ***, **, and * denote significance at the 1%, 5% and 10% levels, respectively.

The empirical findings in this study coincides with those of Stoyanov, Rachev, and Fabozzi (2013). Stoyanov, Rachev, and Fabozzi (2013) studied the sensitivity of the CVaR with respect to tail thickness and the scale of the portfolio return distribution. They concluded that small variations in the tail index do not result in large variations in CVaRs. However, when data are very heavy-tailed, one should backtest CVaR models to ensure the soundness of the model. Empirically, portfolio managers closely monitor daily returns. Daily returns are also prone to skewness and kurtosis problems. A high rebalancing frequency (such as daily or weekly rebalancing) exacerbates this issue. Therefore, a sophisticated mathematical treatment that is capable of working with nonnormal data will yield a rather efficient portfolio, thereby resulting in higher portfolio returns. In other words, the cost of constructing a highly quantitative-oriented model is worthwhile. However, when the rebalancing frequency is extended, the marginal benefits of using complex modeling will diminish as the extent of asymmetry and non-normality is attenuated, making investing in rigorous computations less appealing.

### 5.2 Evaluating Portfolio Sharpe Ratios

The results reported in the previous section reflect the comparisons using the average terms. One problem with averaging is that the reported results are easily biased by outliers. Therefore, a robustness check should be performed to further confirm the empirical findings.

The Sharpe ratio is often considered a tool to compare portfolio performance. Therefore, testing two investment strategies' Sharpe ratios can help researchers identify whether one strategy is better than the alternative. However, given the existence of fat tails in the return data, the usual tests for the differences between the portfolios' Sharpe ratios are inadequate. To conduct a robustness test for portfolio performance, this study adopts the model developed by Ledoit and Wolf (2008) by implementing a studentized time series bootstrap method to make the comparison.

Table 6 reports the results of the Ledoit and Wolf (2008) test for the three portfolio return pairs: GARCH-EVT-Gaussian Copula-CVaR (GAU) versus CVaR, GARCH-EVT-Student's t Copula-CVaR (T) versus CVaR, and, GARCH-EVT-Student's t Copula-CVaR (T) versus GARCH-EVT-Gaussian Copula-CVaR (GAU). The test results reveal that the differences in average returns across the different models during the financial crisis, although sizable, are not statistically different at various rebalancing frequencies. For the post-crisis period, the CVaR model and the two GARCH-EVT-Copula-CVaR models significantly differ at the 99% level in most cases. The test also indicates that the differences between the two GARCH-EVT-Copula CVaR models are not statistically different across the various scenarios.

The results from this section are as follows. First, the returns from the two minimized GARCH-EVT-Copula-CVaR models are statistically different from those obtained from the CVaR model for the full sample and post-crisis periods, thus providing justification for applying the GARCH-EVT-Copula-CVaR structure in the post-crisis period at daily and weekly rebalancing frequencies. Second, although the two minimized GARCH-EVT-Copula CVaR models demonstrate substantially better performance than the CVaR model during the financial crisis period across the four rebalancing frequencies, the differences are not statistically significant.

## 6. Conclusions

In this paper, we evaluate the potential benefits of adopting dynamic GARCH-EVT-Copula-CVaR models in portfolio optimization. We attempt to provide portfolio managers with guidance on when sophisticated tail and dependence models may be valuable for investors. In particular, we examine the fitness of this framework under various rebalancing frequencies and market conditions.

Our results indicate that the portfolios using the two minimized GARCH–EVT–Copula-CVaR models outperform those using the CVaR model at daily and weekly rebalancing frequencies. Therefore, if the company's policy is to review and rebalance portfolios over a shorter period (such as daily or weekly), portfolio managers may have an incentive to adopt GARCH-EVT-Copula-CVaR to gain a better portfolio return. The robustness tests further confirm that the empirical findings are statistically significant for periods of expansion. However, GARCH-EVT-Copula-CVaR modeling is not beneficial for portfolios designed to rebalance over longer horizons. Since tail thickness is the key in deciding whether GARCH-EVT-Copula matters to CVaR optimization process, future research may extend to the areas of developing the tail index to quantify the possible influence from the degree of tail thickness to portfolio optimization.

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