Estimation of the Probability of Credit Card Charge-Off in the Presence of Competing Risks

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The primary reasons for credit card charge-off are delinquency and bankruptcy. In this paper, we develop a dependent competing risks model to investigate the determinants of time to delinquency and time to bankruptcy jointly, and to investigate their interdependence. Our results show that delinquency and bankruptcy are sensitive to unemployment rate and some account-specific measures. In addition, delinquency is highly sensitive to recent legislative reform (Credit Card Act of 2009). We find a negative interdependence between the delinquency and bankruptcy hazards through a significant negative correlation between the unobserved heterogeneities. We show that the independence assumption of competing risks would produce biased estimates and lower the predictive accuracy.

JEL classification: C23; C24; C41; G30; G33
Keywords: Credit Card, Charge-Off, Delinquency and Bankruptcy, Un- observed Heterogeneity, Duration Models, Competing Risks.

1. Introduction

One of the main challenges of credit card issuers is to accurately predict the risk of charge-off which is the amount of a loan that an issuer determines as unlikely to be repaid and counts as a loss. The charge-off rate has increased over time and has reached its highest level during the financial crisis. According to Federal Reserve data[2], the charge-off rate has increased from 3.68% in the fourth quarter of 2006 to 6.44% in the fourth quarter of 2008, and has reached its highest level of 10.79% in the second quarter of 2010. The increasing charge-off rate, specifically during the financial crisis, has raised the question of what factors affect the charge-off and how to predict the risk of charge-off. The object of this paper is to answer this question by applying a duration analysis technique using the panel data of credit card of seven US banks over the period of January 2008 to December 2013.

An important feature that has to be taken into consideration in the credit card duration analysis is that there are two main possible reasons for credit card charge-off. The first possible reason of charge-off is delinquency (D)[3], which simply occurs when balances become six billing cycles past due. The second possible reason of charge-off

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[1] The views in this paper are those of author and do not reflect those of the Office of the Comptroller of the Currency or the Department of Treasury.
[2] Data from Federal Reserve: http://www.federalreserve.gov/releases/chargeoff/chgallsa.htm
[3] This is often referred to as "contractual charge-offs".

is bankruptcy (B), which can occur even if the credit card is in good standing. Thus, delinquency is not a prerequisite for bankruptcy or vise-versa. If one does not differentiate between these two types of charge-off and if these charge-off types have different determinants, then this may lead to incorrect inferences.

The existing literature on the duration analysis of the credit card industry focuses on one charge-off reason at a time [see e.g., Gross and Souleles, 2002; Agarwal and Liu, 2003]. By not accounting for both charge-off reasons simultaneously, one can only evaluate the probability of delinquency and the probability of bankruptcy, but not the probability of charge-off. We aim to address this gap in the literature by simultaneously accounting for delinquency and bankruptcy. To this end, we consider a competing risks model with two cause specific hazard functions, one for delinquency and one for bankruptcy. Furthermore, we add to the literature by extending the model to allow both of these cause specific hazard functions to depend on unobserved heterogeneity terms that can be mutually dependent. The cause specific hazard functions are specified to depend on unobserved heterogeneity terms in addition to observable covariates since there might be some unobservable account-specific characteristics that affect both delinquency and bankruptcy processes. In addition, the dependency between the unobserved heterogeneity terms is taken into account since delinquency and bankruptcy can be a substitute of each other. For example, if there is a high cost of bankruptcy, such as legal and social costs, a credit card holder might consider delinquency as an option in order to avoid bankruptcy. Such behavior implies that higher risk of delinquency might lead to lower risk of bankruptcy, and vice-versa. This may induce a negative correlation between delinquency and bankruptcy processes[4].

In this paper, we assume that each cause specific hazard function has a mixed proportional hazard specification with cause specific baseline hazard, unobserved heterogeneity term, and covariates. The cause specific baseline hazard function for each charge-off type is assumed to follow an expo-power distribution [Saha and Hilton, 1997]. The joint distribution of unobserved heterogeneity terms, is supposed to be bivariate discrete with two admissible values for each heterogeneity term. The choice of the bivariate discrete distribution is motivated by the fact that it is flexible, computationally tractable [Van den Berg, 2001] and suitable for segmentation [see the

[4] It is noteworthy to point out that many empirical competing risks studies assume that the cause specific hazard functions are independent conditional on the observed covariates [see e.g., Duffie *et al*, 2005]. That is to say, when estimating one cause specific hazard function, other causes are treated as censored observations. In addition to computational convenience, one of the main reasons for these studies to make the independence assumption is the common misunderstanding that dependent competing risks specifications are not identifiable. This non-identifiability property is studied in detail by Tsiatis (1975), who proves that for any joint survival function with arbitrary dependence between the competing risks, one can find a different joint survival function with independent competing risks. If that is the case, then there is no point to complicate the model with dependence assumption because the data cannot test for it anyway. However, Tsiatis (1975)'s argument is valid only if the sample is homogenous. Thus, the problem of non-identifiability can be resolved by introducing heterogeneity through the variation of the observed covariates, as discussed in length by Heckman and Honore (1989), Abbring and van den Berg (2003), and Colby and Rilstone (2004).

discussion in Wedel *et al*, 1999]. A set of, potentially time varying, covariates are considered as they are expected to affect both probabilities of delinquency and bankruptcy.

Based on the estimated parameters of the bivariate discrete distribution, we find a significant negative correlation between the unobserved heterogeneity terms. This finding indicates that accounts with relatively high probability of charge-off due to delinquency have a lower probability of charge-off due to bankruptcy, and vise-versa. Regarding the effects of the covariates, our results show that Unemployment Rate, Credit Card Act of 2009, and some account-specific measures (including Behavioral Score, FICO Score, Payment, and Balance) have a significant effect on the specific hazard function of delinquency. In addition, FICO Score, Balance, and Unemployment Rate have a significant effect on the specific hazard function of bankruptcy.

The remainder of this paper is organized as follows. Section 2 reviews the related studies. Section 3 presents the data and some descriptive statistics. Section 4 presents the model. Section 5 reports the empirical results. Section 6 reports the predicted results for few credit card accounts based on the estimated model. Section 7 concludes.

2. Related Studies

There are few empirical studies using duration models to assess the relative importance of different variables in predicting delinquency and/or bankruptcy. To begin, Gross and Souleles (2002) employ a dynamic probit model[5] to predict the probabilities of delinquency and bankruptcy. The authors use a panel dataset of credit card accounts, which are representative of all open accounts in 1995. In their study, the explanatory variables control for time effects, account age, measures of account risk, and the impact of macroeconomic conditions. Gross and Souleles (2002) conclude that age, risk, and macro factors have significant effects on delinquency and bankruptcy. However, once they control for all of the above effects as well as state dummies, they do not find unemployment to have a significant effect on delinquency and bankruptcy.

Agarwal and Liu (2003) use the same econometric framework as Gross and Souleles (2002) to assess the effect of unemployment on delinquency. The authors note that previous empirical studies, including Gross and Souleles (2002), did not consistently find a significant effect of unemployment on bankruptcy and delinquency. Agarwal and Liu (2003) argue that the lack of a significant effect of unemployment may be due to the fact that the sample periods of those studies do not have enough variation in unemployment. Using a panel dataset that covers around 700 thousand accounts over the period 1995-2001, the authors find conclusive evidence of a significant effect of unemployment on delinquency. As for the other

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[5] As shown by Shumway (1998), dynamic probit models are equivalent to discrete duration models.

explanatory variables used in their study, the results are largely in line with those of Gross and Souleles (2002).

Bellotti and Crook (2008) use a Cox proportional hazards model to examine the effects of macroeconomic variables such as bank interest rates, unemployment index, and house price on probability of delinquency. The authors argue that these macroeconomic variables cannot readily be included in logistic regression models. They use a sample of credit card accounts provided by a UK bank between 1997 and 2001. Bellotti and Crook (2008) show that a more accurate prediction of the probability of delinquency can be achieved by using a Cox proportional hazards model which include macroeconomic variables instead of using a logistic regression model which omits them.

Banerjee and Canals-Cerda (2012) assume that an account can be in one of the several current or nonpayment states at each particular point in time. They propose a dynamic multinomial logit framework to predict the probability of delinquency. At each point in time, the probability of delinquency is considered to be a function of account characteristics, customer characteristics, economic environment, and past nonpayment history up to the present time. They use a panel dataset of credit card accounts from a credit bureau over the period 2005 to 2010. Banerjee and Canals-Cerda (2012) find that unemployment plays a significant role in the probability of transition across nonpayment states in general and the probability of delinquency in particular.

3. Data Description and Summary Statistics

A panel dataset of individual credit card accounts obtained from seven US banks is used for our empirical analysis. These seven banks are some of the largest credit card issuers in the United States; hence the dataset should be representative of individual credit card accounts in the United States.

The dataset essentially contains everything that the issuers know about their accounts. In particular, the dataset includes information on account origination date, credit card holder's income at origination, monthly credit limit, FICO score at origination, internal monthly behavior score, monthly purchase, monthly balance, monthly payment, and others. One of the noteworthy features of this database is that it indicates whether or not an account is charged-off in the current month and, if it is, it provides a reason for charge-off; these charge-off reasons include: delinquency, bankruptcy, deceased and other.

The focus here is on general-purpose unsecured credit cards[6] that are originated from January 2008 onwards. The reason to consider only credit cards with an origination date on or after January 2008 is the following. For credit cards with

[6] General purpose credit cards (such as MasterCard, Visa, American Express, and Discover) are those that are intended for general use by a credit card holder, and are not associated with a single merchant or a limited-usage. Unsecured credit cards are those that are backed only by the promise of the credit card holder to pay accumulated charges and interest.

origination dates prior to January 2008, the dataset contains records only on the accounts that were active on or after January 2008. Thus, the dataset suffers from survivorship bias pre January 2008. This bias could adversely affect the reliability of the results. So, the credit cards that were originated before January 2008 are excluded from this study.

As of the end of December 2013, there are 152,720,006 individual credit card accounts in our sample. Out of these accounts, we first exclude accounts that the state of their billing address is not one of the 50 states or the District of Columbia. Second, we exclude accounts with missing value for charge-off reason or current charge-off flag. Third, we remove accounts with missing values or out-of-range values for FICO score at origination[7]. Forth, we remove accounts with (i) even one missing value or (ii) all zero values for monthly credit limit, internal monthly behavior score, monthly purchase, monthly balance, or monthly payment[8]. We finally drop accounts with a flag indicating charge-off due to death, charge-off due to other, involuntarily frozen account, closed/revoked account, closed due to borrowers request, frozen due to potential fraud, or account sold[9]. After applying these criteria, the sample size consists of 18,753,069 accounts. Since the number of accounts is very large, we randomly sample 5% of the accounts[10]. Thus, our final sample size is 937,654 out of which 914,668 are active, 17,868 are charged-off due to delinquency, and 5,118 are charged-off due to bankruptcy. The final sample is merged with monthly data on unemployment rate obtained from Bureau of Labor Statistics.

3.1 Individual Credit Card Account Data

The lifetime (age) of a charged-off account is measured as the difference in months between the origination date and the charge-off date. Since an active account is operating as of December 2013, its lifetime is measured as the difference in months between the origination date and December 2013. Table 1 provides the statistics for lifetimes of all accounts, active accounts, charged-off accounts, and subcategories of charged-off accounts. For all accounts, the mean lifetime is about 23 months with a median of 19 months. The mean value is close to that reported by Gross and Souleles (2002). The mean and median lifetimes are similar for active accounts and charged-off accounts. Specifically, active accounts have a mean and median of 23.17 months and 19 months, respectively, and charged-off accounts have a mean and median of 22.26 months and 20 months, respectively. The mean and

7 FICO score values less than 300 or greater than 850 are considered as out-of-range values.

8 For some accounts, negative values for monthly payment are observed. These accounts are also excluded from our analysis.

9 For some accounts that are not active as of December 2013, the dataset does not provide any reasoning. These accounts are also excluded from our analysis.

10 Summary statistics of the 5% sample align well with summary statistics of the entire sample, and are available upon request.

median lifetimes are quite similar for the charged-off subcategories, with charged-off accounts due to delinquency having the mean and median of 22.42 months and 20 months, respectively, and charged-off accounts due to bankruptcy having the mean and median of 21.69 months and 19 months, respectively. In addition, the values of 25th quantile are comparable among the charged-off subcategories, as well as the values for 75th quantile. This indicates that, regardless of the charged-off reason, 25% and 75% of accounts are charged-off approximately within 13.5 and 28.5 months, respectively.

Table 1: Statistics for Lifetimes of Individual Credit Card Account by Status

Status	Mean	Median	Sd	Min	Max	Q(25%)	Q(75%)
All Accounts	23.14	19.00	16.12	1.00	72.00	10.00	33.00
Active Accounts	23.17	19.00	16.22	1.00	72.00	10.00	33.00
Charged-off Accounts	22.26	20.00	11.40	2.00	72.00	14.00	29.00
Delinquency	22.42	20.00	11.22	2.00	72.00	14.00	29.00
Bankruptcy	21.69	19.00	12.01	2.00	71.00	13.00	28.00

Notes: The table provides the descriptive statistics for lifetimes (in months) of all accounts, active accounts, charged-off accounts, and subcategories of charged-off accounts from January 2008 to December 2013. The descriptive statistics include the mean, median, standard deviation, minimum, maximum, 25% quartile, and 75% quartile.

Table 2: Charge-off Rates (January 2008 - December 2013)

Year	2008	2009	2010	2011	2012	2013
# of Accounts at the Start of the Year	0	35,603	109,075	216,414	382,621	645,616
# of Originated Accounts During the Year	35,671	74,728	110,028	169,524	268,264	279,439
# of Charged-off Accounts Due to Delinquency During the Year	31	907	1,971	2,463	4,091	8,405
# of Charged-off Accounts Due to Bankruptcy During the Year	37	349	718	854	1,178	1,982
Charge-off Rate Due to Delinquency	—	2.55%	1.81%	1.14%	1.07%	1.30%
Charge-off Rate Due to Bankruptcy	—	0.98%	0.66%	0.39%	0.31%	0.31%
Charge-off Rate	—	3.53%	2.47%	1.53%	1.38%	1.61%

Notes: The table provides the number of accounts at the start of the year, number of originated accounts during the year, and number of charged-off accounts due to delinquency and bankruptcy during the year. This table also provides the charge-off rates due to delinquency, charge-off rates due to bankruptcy and total charge-off rates at a given time.

Table 2 shows the charge-off rates due to delinquency, charge-off rates due to bankruptcy, and total charge-off rates at a given time without considering the age of the account. Also included in Table 2 are the number of accounts at the start of the year[11], number of originated accounts during the year, and number of charged-off accounts due to delinquency and bankruptcy during the year. The number of originated accounts increased from 74,728 in 2009 to 279,439 in 2013, an annual increase of about 68.48%. The number of charged-off accounts due to delinquency and bankruptcy also increased from 2009 to 2013. In particular, the number of charged-off accounts due to delinquency increased from 907 in 2009 to 8,405 in 2013 (an annual increase of about 206.67%), and the number of charged-off accounts due to bankruptcy increased from 349 in 2009 to 1,982 in 2013 (an annual increase of about 116.98%). The average annual charge-off rates due to delinquency and bankruptcy are 1.57% and 0.53%, respectively, with the highest values in the year 2009. In fact, the charge-off rates in the year 2009 are more than 1.4 times greater than those charge-off rates in the years following 2009. This result may indicate the impact of the financial crisis on the credit card industry.

Let us now focus on the charge-off rates due to delinquency, charge-off rates due to bankruptcy, and total charge-off rates at a given age without taking time into consideration. To do so, we provide in Figure I the smoothed nonparametric estimate[12] of the hazard functions for all charge-off types (the top panel), for charge-off type due to delinquency (the middle panel), and for charge-off type due to bankruptcy (the bottom panel). In all three panels of Figure I, the probability of charge-off increases during the first few years after origination, peaks around 2 years, and then decreases. In brief, the hazard function appears to be an inverted U-shaped, which is consistent with the study by Gross and Souleles (2002).

3.2 Selection of Explanatory Variables

The determinants of delinquency and bankruptcy hazards as well as their interdependence are examined using the following explanatory variables:

- **Behavioral Score[t]** reports monthly internal score provided by issuers.
- **FICO Score** reports FICO score at origination.
- **Limit [t]** reports monthly amount of the credit line.
- **Purchase [t]** reports monthly purchase during the cycle.
- **Payment[t]** reports monthly amount of all payment received during the cycle.
- **Balance [t]** reports monthly outstanding balance at the end of the cycle.

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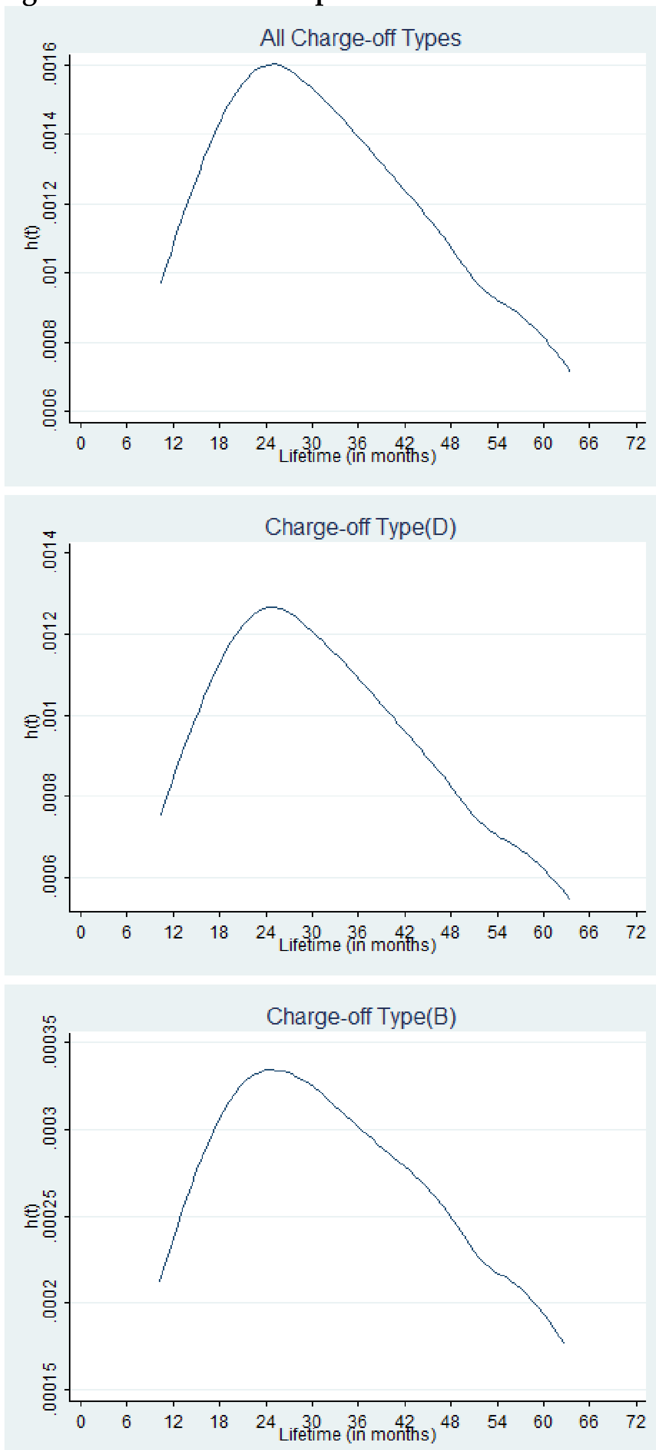
[11] The number of accounts at the start of 2008 is zero, as our sample begins in January 2008.

[12] The estimate is based on the Nelson-Aalen estimator [see Nelson, 1972; Aalen, 1978]. To smooth the Nelson-Aalen estimator, we specify an Epanechnikov kernel function with the default bandwidth in STATA, which is

$$0.9 * \mathrm{Min}\left(\sigma_t, \frac{Interquartile\ Range\ t}{1.349}\right) * n^{-0.2}$$

where σ is the standard deviation of ages (t) and n is the total number of accounts.

Figure I: Smoothed Nonparametric Hazard Function



Notes: The figure displays the smoothed nonparametric estimate of the hazard functions for all charge-off types in the top panel, for charge-off type (D) in the middle panel, and for charge-off type (B) in the bottom panel. The estimate is based on the Nelson-Aalen estimator.

Table 3: Statistics of Time-Varying Explanatory Variables of Individual Credit Card Accounts by Status

			Behavioral Score[t]			
Status	Mean	Std.Dev.	Min	Max	Skewness	Excess Kurtosis
All Accounts	511.11	167.26	157.4	672.19	-0.96	0.63
Active Accounts	536.83	133.58	258.35	667.81	-1.09	2.24
Charged-off Accounts	438.33	287.08	28.09	771.1	-0.46	-3.1
Delinquency	435.13	289.02	25.52	773.73	-0.39	-3.45
Bankruptcy	449.52	280.31	37.05	761.91	-0.73	-1.9
			Limit[t]			
All Accounts	6,435.66	323.7	6,017.99	6,824.18	0.03	-0.38
Active Accounts	6,505.34	312.37	6,112.58	6,867.70	0.01	-0.57
Charged-off Accounts	4,024.68	307.93	3,687.32	4,410.88	0.04	-0.87
Delinquency	3,740.18	287.74	3,425.70	4,105.67	0.05	-0.83
Bankruptcy	5,017.92	378.42	4,600.68	5,476.40	0.00	-1.03
			Purchase[t]			
All Accounts	448.43	519.99	-0.31	1,920.37	2.41	6.17
Active Accounts	480.3	471.41	76.74	1,827.73	2.13	5.09
Charged-off Accounts	168.08	374.67	-23.41	1,435.87	2.76	6.46
Delinquency	156.91	364.11	-26.14	1,407.54	2.80	6.71
Bankruptcy	207.08	411.54	-13.87	1,534.81	2.62	5.59
			Payment[t]			
All Accounts	374.23	451	2.65	1,760.61	2.07	4.86
Active Accounts	418.10	446.89	29.74	1,786.13	1.94	4.87
Charged-off Accounts	87.00	155.23	0.41	657.73	2.01	3.01
Delinquency	81.46	151.43	0.00	638.38	2.12	3.23
Bankruptcy	106.35	168.49	1.85	725.30	1.65	2.22
			Balance[t]			
All Accounts	1,339.09	872.32	50.71	2,840.37	0.86	2.28
Active Accounts	1,405.26	750.56	375.24	2,780.53	0.83	1.38
Charged-off Accounts	2,906.10	1,112.80	679.36	4,064.09	-0.92	-1.46
Delinquency	2,783.39	1,070.09	633.49	3,907.17	-0.91	-1.55
Bankruptcy	3,344.53	1,261.89	839.49	4,611.95	-0.95	-1.15

The table provides the descriptive statistics for monthly Behavioral Score, Limit, Purchase, Payment, and Balance of all accounts, active accounts, charged-off accounts, and subcategories of charged-off accounts from January 2008 to December 2013. The descriptive statistics include the mean, standard deviation, minimum, maximum, skewness, and excess kurtosis; The reported descriptive statistics are averaged over accounts.

- **Unemployment rate [t-6]** reports seasonally adjusted monthly unemployment rate lagged by 6 months in the state of billing address.

It is of importance to note that a significant regulatory change took place during our sample period: the Credit Card Act of 2009. Such changes in regulation might affect the delinquency and bankruptcy hazards, and so the following explanatory variable is also included:

- **Credit Card Act** reports a dummy variable equal to 1 for the months after the Credit Card Act of 2009 became fully effective (i.e. February 2010), 0 otherwise.

The set of explanatory variables is divided into the subsets of time-varying and time-invariant variables, and their summary statistics are reported in tables 3 and 4, respectively, for all accounts, active accounts, charged-off accounts, and sub-categories of charged-off accounts. In table 3, the characteristics of the monthly Behavioral Score, Limit, Purchase, Payment, and Balance averaged over accounts are reported. The measures of average skewness and average kurtosis indicate that the distributions of the monthly Behavioral Score, Limit, Purchase, Payment, and Balance are not normal. In addition, the average means of Behavioral Scores, Limit, Purchase, and Payment for active accounts are higher than for charged-off accounts. However, the average mean of balance is lower for active accounts than for charged-off accounts. Finally, the average characteristics are different between the subcategories of charged-off accounts. This suggests that the subcategories of charged-off accounts are not homogeneous in their average characteristics of the monthly Behavioral Score, Limit, Purchase, Payment, and Balance. Thus, it might be misleading if one does not differentiate between the two types of charge-off.

Table 4: Statistics of Time-Invariant Explanatory Variables of Individual Credit Card Accounts by Status

Status	Mean	Median	Sd	Min	Max	Q(25%)	Q(75%)
All Accounts	740.51	746.00	62.32	353.00	850.00	696.00	792.00
Active Accounts	741.85	748.00	61.80	353.00	850.00	698.00	793.00
Charged-off Accounts	687.43	691.00	59.90	406.00	850.00	656.00	727.00
Delinquency	682.93	686.00	61.51	406.00	850.00	651.00	724.00
Bankruptcy	703.11	706.00	50.88	407.00	847.00	675.00	737.00

Notes: The table provides the descriptive statistics for FICO Score of all accounts, active accounts, charged-off accounts, and subcategories of charged-off accounts from January 2008 to December 2013. The descriptive statistics include the mean, median, standard deviation, minimum, maximum, 25% quartile, and 75% quartile.

4. Econometric Methodology

In this section, we propose a dependent competing risks duration model which is capable to incorporate time-varying covariates and censored observations easily. More importantly, the model controls for unobserved covariates, allows for estimating the delinquency and bankruptcy hazards jointly, and accounts for the interdependence of these hazards. We start by describing the specification of the model, and then derive the likelihood function.

4.1 Model Specification

As mentioned in the introduction, there are two types of credit card charge-off: delinquency (D) and bankruptcy (B). An individual credit card account can experience both delinquency and bankruptcy multiple times during its lifetime. To simplify our analysis, we limit our attention to the first time that an individual account experiences charge-off.

Let the nonnegative random variables T_D and T_B be the potential lifetimes until delinquency (D) and bankruptcy (B), respectively. In the competing risks framework, one only observes one lifetime, namely the minimum one, $T = \min[T_D, T_B]$, and the corresponding actual charge-off type, $J \in \{D, B\}$.

Let us now consider a vector of observable covariates at age t, denoted as $x(t)$, and a vector of unobservable covariates, denoted as $v = (v_D, v_B)$. The observable covariates are considered as they are expected to affect both probabilities of charge-off due to delinquency (D) and charge-off due to bankruptcy (B). The unobservable covariates, called unobserved heterogeneity terms or frailties, are also introduced and denoted by v_D and v_B. v_D captures the unobserved determinants of T_D, and v_B captures the unobserved determinants of T_B. The advantage of introducing two unobserved heterogeneity terms is the possibility of exploring the dependence between T_D and T_B, whenever v_D and v_B are positively or negatively correlated. In this respect, we want to avoid using the too restrictive one factor model in which $v_D = a_D w$ and $v_B = a_B w$ depend on a unique underlying heterogeneity, as in Flinn and Heckman (1982), Clayton and Cuzick (1985), and Heckman and Walker (1990). The advantage of our model is that it does not a priori restrict the sign of dependence if a sufficiently flexible class of joint distributions is chosen for the unobserved heterogeneity terms.

In the following regularity assumptions, index i, $i = 1, \ldots, n$, denotes individual credit card accounts:

Assumption A.1: a) The unobserved heterogeneities are time invariant, and depend on the individual credit card accounts i. b) The individual heterogeneities (v_{iD}, v_{iB}), $i = 1, \ldots, n$, are independent, and have the same distribution $G(v_D, v_B)$.

Assumption A.2: The potential lifetimes T_{iD} and T_{iB}, $i = 1, \ldots, n$, are independent conditional on the observable covariate histories $X_i = \{x_i(t), t \in N\}$, $i = 1, \ldots, n$, and on heterogeneities (v_{iD}, v_{iB}), $i = 1, \ldots, n$.

Assumption A.3: The individual heterogeneities are independent of the covariate histories.

Assumption A.4: a) The variables T_{iD} (resp. T_{iB}), $i = 1, \ldots, n$, have identical conditional distributions given the individual covariate histories and the individual unobserved heterogeneities.

b) The conditional distribution of T_{iD} (resp. T_{iB}) given the individual covariate histories and the individual unobserved heterogeneities depends on the associated individual covariate history X_i and specific individual heterogeneity v_{iD} (resp. v_{iB}).

c) The cause specific hazard functions at age t depend on individual covariate history by means of $x_i(t)$ only.

Assumption A.1 is commonly imposed in microeconomic studies and it indicates that the focus of the analysis is on individual omitted heterogeneity. It implies that individual heterogeneities that depend on both individual accounts and time are excluded. This allows us to assume away the moral hazard phenomena [see e.g., Gourieroux and Jasiak, 2004] and the omitted dynamic variables[13].

Assumptions A.2 to A.4 are standard. Under Assumptions A.1-A.4, the overall hazard function for individual credit card account *i* conditional on both the observed covariates and unobserved heterogeneities is:

$$h(t; x(t), v_D, v_B) = \lim_{\Delta t \to 0} \frac{\Pr[t \leq T < t + \Delta t \mid T \geq t, x(t), v_D, v_B]}{\Delta t} \tag{4.1}$$

and the cause specific hazard function for individual credit card account *i* conditional on both the observed covariates and unobserved heterogeneity for charge-off type j is :

$$h_j\big(t; x(t), v_j\big) = \lim_{\Delta t \to 0} \frac{\Pr[t \leq T < t + \Delta t, J = j \mid T \geq t, x(t), v_j]}{\Delta t}, \qquad j = D, B \tag{4.2}$$

where the individual index *i* is suppressed for ease of exposition. Given that only one of the charge-off types can occur, the overall hazard function is the sum of the specific hazard functions for charge-off type (D) and charge-off type (B)[14], i.e.,

$$h(t; x(t), v_D, v_B) = h_D(t; x(t), v_D) + h_B(t; x(t), v_B) \tag{4.3}$$

As the heterogeneity terms are unobserved, the hazard functions given above are stochastic. The stochastic intensities are not "observable". The distribution of the duration conditional on the observable covariates is only derived by integrating out

[13] The omitted time dependent variables could be account specific, such as the volatility of credit card spending, or common to all accounts, such as variables representing contagion or systemic risk. The analysis of these unobserved variables is left for further research.

[14] There is a possibility that the charge-off type (D) and the charge-off type (B) occur simultaneously. In this situation, the combination of the two can be defined as a new type of charge-off [see Kalbfleisch and Prentice, 2002, page 251]. We have not, however, encountered this situation in our dataset.

the unobservable components. In particular, if the individual heterogeneities v_{iD} and v_{iB} are dependent, integrating will create dependence between the lifetimes T_{iD} and T_{iB}, and also between the two specific stochastic intensities. For ease of exposition, the index i is suppressed in the formulas below.

Assumption A.5: The cause specific hazard functions conditional on $(x_i(t), v_iD, v_iB)$, $i = 1, \ldots, n$, are mixed proportional hazard functions :

$$h_D(t; x(t), v_D) = h_{0D}(t)\exp(x(t)'\beta_D)\exp(v_D)$$
$$h_B(t; x(t), v_B) = h_{0B}(t)\exp(x(t)'\beta_B)\exp(v_B) \tag{4.4}$$

where β_j is the vector of unknown coefficients for the jth charge-off type, $j = D$, B. h_{0D} and h_{0B} are the specific baseline hazard functions for charge-off type (D) and charge-off type (B), respectively.

This mixed proportional hazard (MPH) model can be analyzed either parametrically or semi-parametrically. Since our interest is to predict the risk of charge-off we follow the parametric approach under which both the baseline hazard function and the heterogeneity distribution are parametric.

Assumption A.6: The baseline hazard functions follow an expo-power distribution:

$$h_{0j}(t) = \alpha_j t^{\alpha_{j-1}} \exp(\theta_j t^{\alpha_j}) \tag{4.5}$$

where $j = D, B, a_j > 0, -\infty < \theta_j < +\infty$.

This parametric specification was introduced by Saha and Hilton (1997). It can represent a variety of patterns of the hazard function, including constant, monotonically increasing, monotonically decreasing, U-shaped, inverted U-shaped, or humped-shaped. It includes as a special case the Weibull hazard function for $\theta = 0$ which is monotone. For $\theta \neq 0$, the hazard function has a turning point at $[(1 - a_j)/(a_j\theta_j)]^{1/a_j}$.

Conditional on the observable covariate histories, the distributions of the uncensored and right censored account durations are characterized by the probabilities
$Pr(t \leq T < t + \Delta t, J = j \mid X(t))$ and $Pr(T > c \mid X(c))$, respectively. These probabilities are obtained by integrating out v_D and v_B as:

$$Pr(t \leq T < t + \Delta t, J = j|X(t)) = \int_{vD}\int_{vB} Pr(t \leq T < t + \Delta t, J = j|X(t), v_D, v_B)\,dG(v_D, v_B)$$

$$
\begin{aligned}
& where\ Pr(t \le T < t + \Delta t, J = j | X(t), v_D, v_B) \\
&= Pr(t \le T < t + \Delta t, J = j | T \ge t, x(t), v_j) Pr(T \ge t | X(t), v_D, v_B) \\
&= Pr(t \le T < t + \Delta t, J = j | T \ge t, x(t), v_j) * Pr(T_D \ge t | X(t), v_D) \\
&* Pr(T_B \ge t | X(t), v_B) \\
&= h_j(t; x(t), v_j) \\
&* \exp\left(-\int_0^t h_D(u; x(u), v_D) du\right) \exp\left(-\int_0^t h_B(u; x(u), v_B) du\right) \Delta t \qquad (4.6)
\end{aligned}
$$

where j=D, B. This quantity depends on the covariate histories up to age t only. Moreover, we have:

$$
Pr(T > c | X(c)) = \int_{vD} \int_{vB} Pr(T > c | X(c), v_D, v_B) dG(v_D, v_B)
$$

$$
\begin{aligned}
& where\ Pr(T > c | X(c), v_D, v_B) = Pr(T_D > c | x(c), v_D) Pr(T_B > c | x(c), v_B) \\
&= \exp\left(-\int_0^c h_D(u; x(u), v_D) du\right) \exp\left(-\int_0^c h_B(u; x(u), v_B) du\right). \qquad (4.7)
\end{aligned}
$$

This quantity depends on the covariate history up to age c only.

In practice, the model has to be completed by specifying the joint distribution of the unobserved heterogeneities. In this subsection, we use an extension of the Heckman and Singer (1984) approach [see also Nickell, 1979; Van den Berg *et al*, 2004] and assume the following:

Assumption A.7: The joint distribution of the unobserved heterogeneity terms is bivariate discrete in which v_D and v_B can only take two values. Let v_D^1 and v_D^2 denote the values of v_D, and v_B^1 and v_B^2 denote the values of v_B.

$$v_D^1 > v_D^2 \text{ and } v_B^1 > v_B^2$$

The choice of the discrete heterogeneity is motivated by the segmentation of individual credit card accounts into risky and less-risky accounts, conditional on the covariate histories[15]. Each unobserved heterogeneity can take either a large, or a small value, which has direct effects on the value of intensity. The idea underlying the large and small possible values of heterogeneity resembles the setup with two possible levels of risk changes: up and down, as introduced for instance in the binomial tree for option pricing by Cox et al (1979). Thus, conditional on covariate histories, the set of individual credit card accounts can be divided into four classes that correspond to (v_D^1, v_B^1), (v_D^1, v_B^2), (v_D^2, v_B^1), and (v_D^2, v_B^2), respectively. The sizes of these classes are unknown a priori and will be approximated by means of their associated probability estimates.

Under Assumption A.7, the joint distribution of v_D, v_B is characterized by the following elementary probabilities:

---

15 In the literature on duration data the gamma distribution of unobserved heterogeneity is also used [see e.g., the discussion in Abbring and Van den Berg, 2007].

$$\Pr(v_D = v_D^1, v_B = v_B^1) = p_{11}, \qquad \Pr(v_D = v_D^1, v_B = v_B^2) = p_{12}$$
$$\Pr(v_D = v_D^2, v_B = v_B^1) = p_{21}, \qquad \Pr(v_D = v_D^2, v_B = v_B^2) = p_{22}.$$

with $0 \leq pkl \leq 1$ and $\sum_{k=1}^{2}\sum_{l=1}^{2} pkl = 1$ for k, l = 1,2[16].

The covariance of v_D and v_B can be derived as (see Van den Berg *et al*, 1994):

$$\mathrm{Cov}(v_D, v_B) = (p_{11}p_{22} - p_{12}p_{21})(v_D^1 - v_D^2)(v_B^1 - v_B^2).$$

and so the correlation between v_D and v_B becomes:

$$\rho(v_D, v_B) = \frac{(p_{11}p_{22} - p_{12}p_{21})}{\sqrt{(p_{11}+p_{12})\,(p_{11}+p_{21})(p_{22}+p_{12})\,(p_{22}+p_{21})}}.$$

Variables v_D and v_B will be perfectly correlated if either $p_{12} = p_{21} = 0$, or $p_{11} = p_{22} = 0$. Further, v_D and v_B are independent, if and only if $p_{11}p_{22} - p_{12}p_{21} = 0$.

As mentioned in the introduction, the effect of the unobserved heterogeneity is expected to be twofold: First, it can correct inpart for the estimation bias in the sensitivity coefficients of the covariates and in the age pattern of the hazard function. These effects will be accommodated by the marginal distributions of both heterogeneities, i.e. by the parameters $p_{11} + p_{12}$ and $p_{21} + p_{22}$ for v_D, and by the parameters $p_{11} + p_{21}$ and $p_{12} + p_{22}$ for v_B. Second, it can create dependence between the durations by means of the correlation coefficient ρ[17].

Under Assumption A.7, the characteristics of the uncensored and right censored distributions of account lifetimes become:

$$Pr(t \leq T < t + \Delta t, J = j|X(t)) = \sum_{k=1}^{2}\sum_{l=1}^{2} Pr\left(t \leq T < t + \Delta t, J = j\middle|X(t), v_D^k, v_B^l\right) p_{kl} \quad (4.8)$$

and

$$Pr(T > c\,|X(c)) = \sum_{k=1}^{2}\sum_{l=1}^{2} Pr\left(T > c\middle|X(c), v_D^k, v_B^l\right) p_{kl} \quad (4.9)$$

4.2 The Likelihood Function

Let us now consider a sample of possibly right-censored credit card accounts lifetimes variables for accounts $i, i = 1, \ldots, n$.

The unknown parameters are:

$\beta = (\beta_D, \beta_B)$, which defines the sensitivity of the cause specific hazard functions for each charge-off type to the observable covariates,

---

[16] To ensure that the probabilities lie between [0,1] and sum up to 1, we apply the logistic transformation, i.e., $p_{kl} = \frac{\exp(q_{kl})}{\sum_{k=1}^{2}\sum_{l=1}^{2}\exp(q_{kl})}$ where $-\infty < q_{kl} < +\infty$, for k, l = 1,2.

[17] This double effect cannot be accounted for by a parametric copula written on the two duration variables (see e.g., Gregoriou and Pascalau, 2012].

a = (a_D, a_B) and θ = (θ_D, θ_B), which characterize the expo-power baseline hazard functions for each charge-off type,

v= (v_D^1, v_D^2, v_B^1, v_B^2), which defines the admissible values of unobserved individual heterogeneities with $v_D^1 > v_D^2$, $v_B^1 > v_B^2$

and p = (p_{11}, p_{12}, p_{21}, p_{22}), p_{ij} ≥ 0, $p_{11} + p_{12} + p_{21} + p_{22}$ = 1, which provides the associated elementary probabilities.

In order to avoid identification problems, we assume no constant covariate that is no intercept in the proportionality term. The levels of the intensities are captured by means of the values v_D^1, v_D^2, v_B^1, v_B^2, which are left unconstrained.

We derive the expression of the likelihood function:

$$l(\beta,\propto,\theta,v,\rho)$$

$$\propto \prod_{i\in W_{11}}\left[\sum_{K=1}^{2}\sum_{K=1}^{2}\left(h_D\big(t_i;x_i(t_i),v_D^k\big)\exp\left(-\int_0^{t_i}h_D\big(u;x_i(u),v_D^k\big)du\right)\exp\left(-\int_0^{t_i}h_B\big(u;x_i(u),v_B^k\big)du\right)_{Pkl}\right)\right]$$

$$*\prod_{i\in W_{12}}\left[\sum_{K=1}^{2}\sum_{K=1}^{2}\left(h_B\big(t_i;x_i(t_i),v_B^k\big)\exp\left(-\int_0^{t_i}h_D\big(u;x_i(u),v_D^k\big)du\right)\exp\left(-\int_0^{t_i}h_B\big(u;x_i(u),v_B^k\big)du\right)_{Pkl}\right)\right]$$

$$*\prod_{i\in W_{2}}\left[\sum_{K=1}^{2}\sum_{K=1}^{2}\exp\left(-\int_0^{C_i}h_D\big(u;x_i(u),v_D^k\big)du\right)\exp\left(-\int_0^{C_i}h_B\big(u;x_i(u),v_B^k\big)du\right)_{Pkl}\ )\right] \quad (4.10)$$

where W_{11} is the set of 17,868 uncensored credit card accounts that are charged- off due to (D), W_{12} is the set of 5,118 uncensored credit card accounts that are charged-off due to (B), and W_2 is the set of 914,668 right censored credit card accounts. This likelihood expression is valid when the covariates are continuously observed since the origination date. This condition is automatically satisfied by covariates x_i, which depend on individual only. But the covariates which depend on time (age) are usually observed in discrete time. In this case, the likelihood function has to be approximated by assuming that the covariates are constant between two consecutive observation dates (The mathematical details of this approximation is available upon request.).

5. Empirical Analysis

Here, we report and discuss the maximum likelihood estimates for three different models. The first model, model (1), is the unrestricted model introduced in the previous section. Model (2) is the model in which the unobserved heterogeneities v_D and v_B are assumed independent. This independence assumption is equivalent to the condition $p_{11}\,p_{22}$- $p_{12}\,p_{21}$=0, whenever $v_D^1 \neq v_D^2$ and $v_B^1 \neq v_B^2$. Under model (2), the two competing risks are independent conditional on the observed covariates. Finally, model (3) is the model without heterogeneity.

Tables 5, 6 and 7 provide estimation results for model (1), model (2), and model (3), respectively [18] . Behavioral Score, Purchase, Payment, and Balance are normalized[19]. The standard errors reported for the α = (αD, αB), p = (p11, p12, p21, p22), and ρ are estimated using the delta method[20]. The intercepts are set equal to zero in all models with unobserved heterogeneities (that are models (1) and (2)) since the intercepts cannot be distinguished from multiplicative constants in unobserved heterogeneities.

Table 5: Dependent Competing Risks Estimates

Model (1)	**Delinquency**			**Bankruptcy**		
	Coeff.	S.E.	Sign.	Coeff.	S.E.	Sign.
Behavioral Score	-0.875	-0.29	***	-0.20	-0.17	
FICO Score	-0.726	-0.37	**	-0.67	-0.35	*
Limit	-0.011	-0.10		-0.01	-0.12	
Purchase	-0.001	-0.15		-0.00	-0.10	
Payment	-0.401	-0.08	***	-0.00	-0.14	
Balance	0.305	-0.10	***	0.20	-0.10	*
Unemployment	0.373	-0.11	***	0.453	-0.11	***
Credit Card Act	-0.470	-0.04	***	-0.66	-0.40	
α	1.715	-0.38	***	2.257	-0.01	***
θ	-2.169	-0.12	***	-2.16	-0.11	***
V_D^1	3.018	-0.17	***			
V_D^2	1.169	-0.19	***			
V_B^1	1.901	-0.27	***			
V_B^2	0.255	-0.15	*			
P_{11}	0.275	-0.03	***			
P_{12}	0.255	-0.03	***			
P_{21}	0.357	-0.03	***			
P_{22}	0.113	-0.01	***			
ρ	-0.119	-0.04	***			
Log-likelihood	-79911.23					

Notes: The table provides the maximum likelihood estimates for model (1), which is the model with dependent unobserved heterogeneities. The numbers in parentheses are the standard errors for the estimated coefficients. The reported standard errors for $\alpha = (\alpha_D, \alpha_B)$, $P = (P_{11}, P_{12}, P_{21}, P_{22})$, and ρ are calculated by the delta method. *, **, and *** indicate that the coefficients are statistically significant at the 10%, 5%, and 1% levels respectively.

[18] In the table, the numbers in parentheses are the standard errors for the estimated coefficients. *, **, and *** indicate that the coefficients are statistically significant at the 10%, 5%, and 1% levels, respectively.

[19] Behavioral Scores are generally derived independently by each bank. Thus, to make Behavioral Scores comparable, we normalized them by dividing the score values of each bank by its maximum score value. Purchase, Payment, and Balance are normalized by dividing them by the Limit.

[20] Derivation of the standard errors of the maximum likelihood estimators of $\alpha = (\alpha_D, \alpha_B)$, $p = (p_{11}, p_{12}, p_{21}, p_{22})$, and ρ is available upon request.

Table 6: Independent Competing Risks Estimates with Unobserved Heterogeneities

Model (1)	Delinquency			Bankruptcy		
	Coeff.	S.E.	Sign.	Coeff.	S.E.	Sign.
Behavioral Score	-0.853	-0.341	**	-0.183	-0.182	
FICO Score	-0.663	-0.348	*	-0.593	-0.343	*
Limit	-0.010	-0.104		-0.006	-0.115	
Purchase	-0.001	-0.173		0.00	-0.106	
Payment	-0.401	-0.128	***	0.00	-0.121	
Balance	0.305	-0.103	***	0.201	-0.103	*
Unemployment	0.378	-0.106	***	0.46	-0.106	***
Credit Card Act	-0.40	-0.332		-0.572	-0.395	
α	1.757	-0.497	***	-0.572	-0.014	***
θ	-2.155	-0.121	***	-2.152	-0.114	***
V_D^1	1.728	-0.132	***			
V_D^2	0.839	-0.232	***			
V_B^1	1.322	-0.201	***			
V_B^2	0.095	-0.197				
P_{11}	0.057	-0.016	***			
P_{12}	0.156	-0.017	***			
P_{21}	0.210	-0.029	***			
P_{22}	0.577	-0.051	***			
ρ	-79917					
Log-likelihood	-0.853	-0.341	**	-0.183	-0.182	

Notes: The table provides the maximum likelihood estimates for model (2), which is the model with independent unobserved heterogeneities. The numbers in parentheses are the standard errors for the estimated coefficients. The reported standard errors for $\alpha = (\alpha_D, \alpha_B)$, and $P = (P_{11}, P_{12}, P_{21}, P_{22})$ are calculated by the delta method. *, **, and *** indicate that the coefficients are statistically significant at the 10%, 5%, and 1% levels respectively.

Below, we first compare the models based on the likelihood ratio statistics, and then focus on the results from the most preferred model specification.

Comparison of Model (2) and Model (3): By comparing models (2) and (3), we can test whether there are unobserved independent heterogeneities in the specific hazard functions for charge-off type (D) and charge-off type (B). The test statistics for the presence of v_D and v_B are independent under the null since the likelihood can be factorized into a product of likelihoods specific of charge-off types (D) and (B). However, the likelihood ratio statistic for the null hypothesis $H_{0D} = \{v_D^1 = v_D^2\}$ and for the null hypothesis $H_{0B} = \{v_B^1 = v_B^2\}$ are non-standard because fewer parameters are identified under the null hypothesis than under the alternative. For instance, the probabilities p_{11}, p_{12}, p_{21}, and p_{22} are not identifiable if $v_D^1 = v_D^2$ and $v_B^1 = v_B^2$. A careful analysis of this problem of test is out of the scope of our analysis and would require either assumptions on the local alternatives of interest, or some prior restrictions on the parameter domain to avoid the difficulties [see e.g. Andrews and Ploberger, 1994]. In the literature, it is generally assumed that a test is conservative if the critical value of the chi-square distribution with 2 degrees of freedom is used. Thus, in our analysis, we compare the likelihood ratio statistic with the critical value of the χ_2^2 distribution. For model (2), the log-likelihood values for charge-off types (D) and (B) are -36491.008 and -17077.500, respectively; and for model (3), the log-likelihood values for charge-off types (D) and (B) are -43669.557 and -19155.066, respectively. The calculated values of the likelihood ratio statistic are larger than the critical value of χ_2^2 at the 5% level of significance. Thus, a significant improvement of model (2) over model (3) is concluded.

Table 7: Independent Competing Risks Estimates without Unobserved Heterogeneities

Model (3)	Delinquency			Bankruptcy		
	Coeff.	S.E.	Sign.	Coeff.	S.E.	Sign.
Intercept	-0.327	-0.217		-0.457	-0.299	
Behavioral Score	-0.776	-0.212	***	-0.163	-0.140	
FICO Score	-0.695	-0.273	**	-0.626	-0.280	**
Limit	-0.009	-0.142		-0.005	-0.113	
Purchase	-0.001	-0.173		0.000	-0.143	
Payment	-0.401	-0.103	***	0.000	-0.103	
Balance	0.303	-0.109	***	0.200	-0.202	
Unemployment	0.369	-0.104	***	0.455	-0.105	***
Credit Card Act	-0.482	-0.250	*	-0.609	-0.321	*
α	1.542	-0.581	***	2.922	-0.014	***
θ	-2.158	-0.115	***	-2.144	-0.107	***
Log-likelihood	-81659					

Notes: The table provides the maximum likelihood estimates for model (3), which is the model without unobserved heterogeneities. The numbers in parentheses are the standard errors for the estimated coefficients. The reported standard errors for $\alpha = (\alpha_D, \alpha_B)$ are calculated by the delta method. *, **, and *** indicate that the coefficients are statistically significant at the 10%, 5%, and 1% levels respectively.

Comparison of Model (1) and Model (2): By comparing models (1) and (2), we can test whether the unobserved heterogeneity terms are dependent. Testing for independence between v_D and v_B is equivalent to test the null hypothesis $H_0 = \{p_{11}p_{22}-p_{12}p_{21}=0\} = \{\rho=0\}$. Under the null hypothesis, the likelihood ratio statistic is distributed as a chi-square with 1 degree of freedom. The calculated value of the likelihood ratio statistic is larger than the critical value of χ_1^2at the 5% level of significance. Hence, we conclude that the unobserved heterogeneity terms are dependent, and model (1) is improved significantly over model (2). Equivalently, we can consider the significance of the maximum likelihood estimate of correlation ρ. Using the entries in Table 5, we reject the null hypothesis ρ=0.

As models (2) and (3) are rejected in favor of the unrestricted model (1), in the following, we focus on the results from model (1).

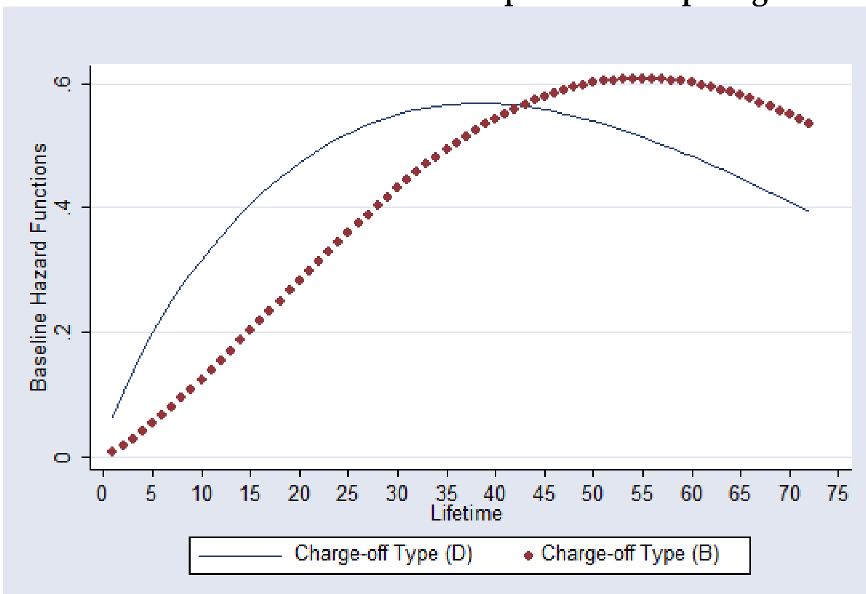
First, we investigate the delinquency hazard. The covariates including Behavioral Score, FICO Score, Payment, and Credit Card Act have a negative significant effect on the specific hazard function for charge-off type (D); in contrast, Balance and Unemployment rate have a positive significant effect on the specific hazard function for charge-off type (D). More specifically, accounts with higher Behavioral Score, higher FICO Score, larger Payment, and smaller Balance are less likely to delinquent. The results are consistent with the finding of Gross and Souleles (2002). In addition, accounts are less likely to delinquent in the presence of the Credit Card Act of 2009. However, accounts are more likely to delinquent in the presence of higher Unemployment Rate, which is consistent with prior research [Gross and Souleles, 2002; Agarwal and Liu, 2003]. The covariates Limit and Purchase appear to have no impact on the specific hazard function for charge-off type (D).

Next, we turn to the bankruptcy hazard. The covariate FICO Score has a negative significant effect on the specific hazard function for charge-off type (B) whereas the covariates Balance and Unemployment rate have a positive significant effect on the specific hazard function for charge-off type (B). More specifically, accounts with higher FICO Score and smaller Balance are less likely to go bankrupt. In addition, accounts become less likely to bankrupt in the presence of lower Unemployment Rate. These findings are consistent with those of Gross and Souleles (2002).

We also plot the specific baseline hazard functions for charge-off type (D) and charge-off type (B), as presented in Figure II, given the estimated parameters of expo-power distribution. Both specific baseline hazard functions show an initial increase, and in fact this increase continues up to the first 38 months. From then onwards, the specific baseline hazard function for charge-off type (D) declines; however, the specific baseline hazard function for charge-off type (B) continues to rise up to 55 months and then begins a gradual decline. These results seem compatible with the finding of Gross and Souleles (2002), although the comparison is difficult. In general, the model used in Gross and Souleles (2002) do not include unobserved heterogeneity. This empirical study reports a hump in the hazard function for charge-off type (D) at about 2 years. We get a larger value of 3 years. This is due to the introduction of unobserved heterogeneity. Indeed, when the heterogeneity is integrated out, or absent as it is common in the standard credit card duration analysis, the negative duration dependence effect is obtained and as a consequence the hump is moved closer to the origin.

Finally, we examine the correlation between v_D and v_B. The estimate of $\rho(v_D, v_B)$ indicates a negative correlation between v_D and v_B, which suggests that accounts with relatively high probability of charge-off due to delinquency (D) have a lower probability of charge-off due to bankruptcy (B), and vise-versa. The test of the null hypothesis $H_0=\{\rho=-1\}=\{p_{11}=p_{22}=0\}$, based on the results in Table 5 indicates that this hypothesis is rejected. This finding confirms that there is no perfect negative correlation between the unobserved heterogeneity terms; and so, conditional on observable covariate histories, there are four classes of accounts that correspond to (v_D^1, v_B^1), (v_D^1, v_B^2), (v_D^2, v_B^1), and (v_D^2, v_B^2), respectively.

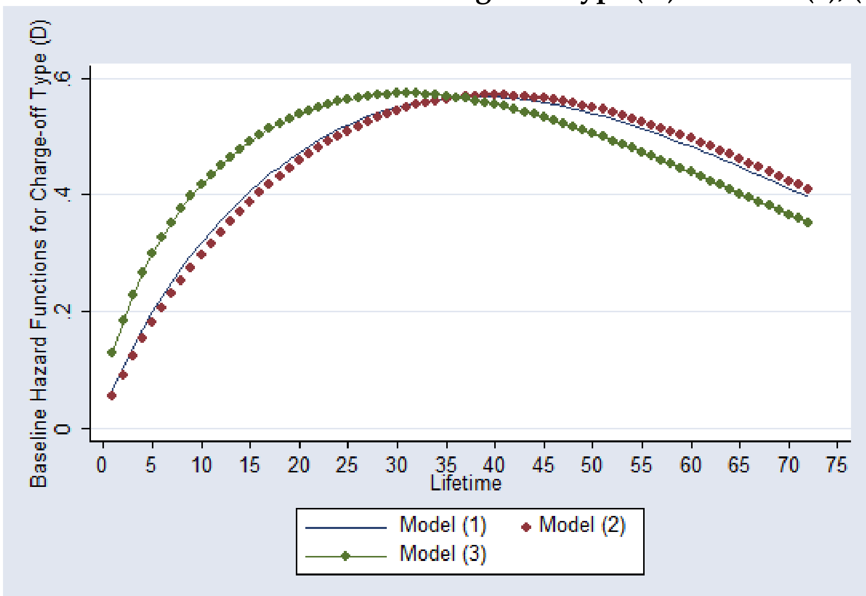
Figure II: Baseline Hazard Functions for a Dependent Competing Risks Model



Notes:The figure displays the estimates of the baseline hazard functions for charge-off type (D) and charge-off type (B). The estimate of the baseline hazard functions for charge-off type j (j=D,B) is obtained using the maximum likelihood estimates of αj and θj (j=D,B) from model (1), and the account lifetimes.

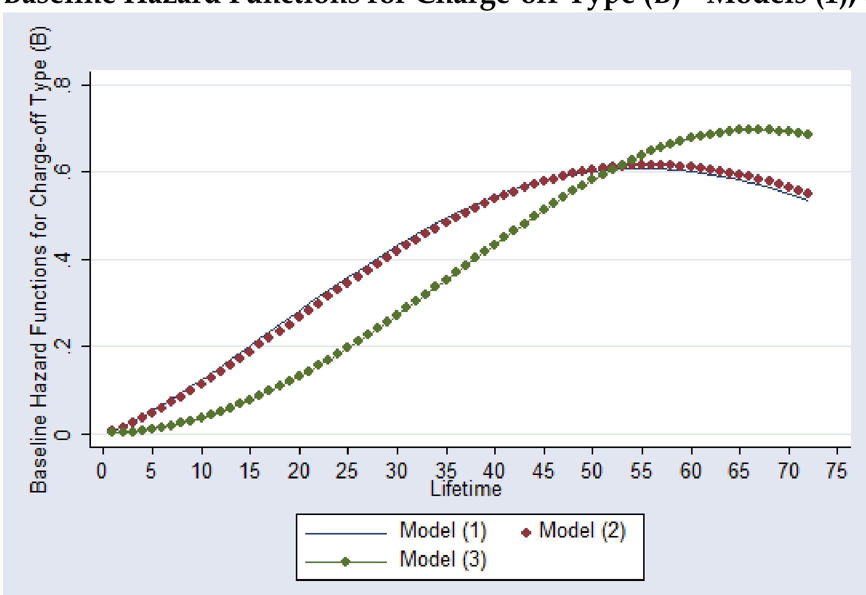
Although we reject models (2) and (3) in favor of model (1), it is still worthwhile to compare the results from those models to model (1) in order to demonstrate the difference. A quick comparison shows that the significance level of the covariates is slightly generally increased in model (1). This is more prominent for the Credit Card Act covariate, in which the estimated coefficient on Credit Card Act for charge-off type (D) is strongly negatively significant in model (1), however, this estimated coefficient is negative and less significant in model (3), and negative and insignificant in model (2). To compare the estimated parameters of expo-power distribution among the three models, the plots of the specific baseline hazard functions for charge-off type (D) and for charge-off type (B) are presented in Figures III and IV, respectively. Figure III shows that for all the three models the specific baseline hazard functions for charge-off type (D) appear to be inverted U-shaped. That is, the probability of delinquency is increasing in the first months, reaches a peak and then decreases. An important observation is that while the specific baseline hazard function for model (1) is roughly equal to the specific baseline hazard function for model (2), it is very different from the specific baseline hazard function for model (3).

Figure III: Baseline Hazard Functions for Charge-off Type (D) - Models (1), (2), and (3)



Notes: The figure compares models (1), (2) and (3) in terms of the estimates of the baseline hazard functions for charge-off type (D). Each estimate is obtained using the maximum likelihood estimates of a_D and θ_D from the corresponding model, and the account lifetimes.

Figure IV: Baseline Hazard Functions for Charge-off Type (B) - Models (1), (2), and (3)



Notes: The figure compares models (1), (2) and (3) in terms of the estimates of the baseline hazard functions for charge-off type (B). Each estimate is obtained using the maximum likelihood estimates of αB and θB from the corresponding model, and the account lifetimes.

In fact, the specific baseline hazard function for model (3) is higher than the specific baseline hazard functions for model (1) and (2) up to about the first 35 months, but it is lower afterwards. This result suggests that the probability of delinquency in the model without unobserved heterogeneities would be overestimated in the first few months, and would be underestimated after that. As shown in Figure IV, in all three models, the probability of bankruptcy increases until it reaches a peak, and then very gradually decreases. Quite noteworthy is that the specific baseline hazard function for model (3) is lower in about the first 53 months compared with the specific baseline hazard functions for models (1) and (2), but it is higher after that. This indicates that the probability of bankruptcy in the model without unobserved heterogeneities would be underestimated in the first few months, and would be overestimated after that.

6. Prediction of the Risk of Charge-Off

The estimated model (1) can be used for predicting the probability of charge-off of an account by a future date, and predicting the probability of the reason of charge-off. To illustrate this, we consider three active accounts as of December 2013 from our database. We observe if the status of these accounts changes between December 2013 and November 2014 as the dataset covers data until November 2014. In particular, we observe that the status of the first account (account (1)) remains active as of November 2014, the status of the second account (account (2)) changes to delinquency in November 2014, and the status of the third account (account (3)) changes to bankruptcy in July 2014. The characteristics of the three accounts are summarized in Table 8.

Table 8: Characteristics of Credit Card Accounts (1), (2), and (3)

	Account (1)	Account (2)	Account (3)
Age	8	50	70
Last Month Behavioral Score	938	493	267
FICO Score	785	731	549
Last Month Limit	$10,000	$7,000	$3,000
Last Month Purchase	$1,191	$2,235	$0
Last Month Payment	$1,660	$0	$102
Last Month Balance	$1,190	$3,866	$2,944
Unemployment Rate	6.8%	8.9%	7.4%
Credit Card Act	Yes	Yes	Yes

Notes: The table lists the characteristics of three credit card accounts selected from the database. Age is reported in months, Unemployment Rate lagged by 6 months from December 2013 is reported, and the presence of a Credit Card Act is reported by “Yes”, “No” otherwise.

For each of these accounts at age t, we calculate the following probabilities where the future values of time dependent covariates are held fixed[21]:

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[21] For each account, the values of the variables in the last month (December 2013) are considered as the constant future values of the respective variables.

- Probability of charge-off by age t+h, given the covariates:

$$Pr\big(T \le t + h\big|X(t+h)\big)$$
$$= 1 - Pr(T > t + h\,|X(t+h))$$
$$= 1 - \sum_{k=1}^{2}\sum_{l=1}^{2}\exp(-H_D\big(t+h; X(t+h), v_D^k\big)) * \exp(-H_B\big(t+h; X(t+h), v_B^l\big))\, p_{kl}$$

- Probability of charge-off by age t+h, given the covariates and that the account is not charged-off until age t:

$$Pr\big(T \le t + h\big|\,T > t, X(t+h)\big)$$
$$= \frac{\Pr\big(T > t\,\big|\,X(t)\big) - \Pr\big(T > t+h\,\big|\,X(t+h)\big)}{\Pr\big(T > t\,\big|\,X(t)\big)}$$
$$= 1 - \frac{\sum_{k=1}^{2}\sum_{l=1}^{2}\exp(-H_D\big(t+h;X(t+h),v_D^k\big)) * \exp(-H_B\big(t+h;X(t+h),v_B^l\big))p_{kl}}{\sum_{k=1}^{2}\sum_{l=1}^{2}\exp(-H_D\big(t;X(t),v_D^k\big)) * \exp(-H_B\big(t;X(t),v_B^l\big))p_{kl}}$$

- Probabilities of charge-off due to delinquency and bankruptcy, given the covariates and that the account is charged-off in the small age interval [t+h, t+h+Δt):

$$Pr(J = D|\; t + h \le T < t + h + \Delta t, X(t+h))$$
$$= \frac{\Pr(\mathrm{t+h} \le \mathrm{T} < \mathrm{t+h} + \Delta\mathrm{t}, \mathrm{J} = \mathrm{D}|\mathrm{X(t+h)})}{\Pr(\mathrm{t+h} \le \mathrm{T} < \mathrm{t+h} + \Delta\mathrm{t}|\mathrm{X(t+h)})} = \frac{\mathrm{A}}{\mathrm{C}}$$

$$Pr(J = B|\; t + h \le T < t + h + \Delta t, X(t+h))$$
$$= \frac{\Pr(\mathrm{t+h} \le \mathrm{T} < \mathrm{t+h} + \Delta\mathrm{t}, \mathrm{J} = \mathrm{B}|\mathrm{X(t+h)})}{\Pr(\mathrm{t+h} \le \mathrm{T} < \mathrm{t+h} + \Delta\mathrm{t}|\mathrm{X(t+h)})} = \frac{\mathrm{B}}{\mathrm{C}}$$

where $\mathrm{A} = \Pr(\mathrm{t+h} \le \mathrm{T} < \mathrm{t+h} + \Delta\mathrm{t}, \mathrm{J} = \mathrm{D}|\mathrm{X(t+h)})$

$$= \sum_{k=1}^{2}\sum_{l=1}^{2} h_D\big(t+h; x(t+h), v_D^k\big) * \exp(-H_D\big(t+h; X(t+h), v_D^k\big))$$
$$* \exp(-H_B\big(t+h; X(t+h), v_B^l\big))\, p_{kl}\Delta t$$

and $\mathrm{B} = \Pr(\mathrm{t+h} \le \mathrm{T} < \mathrm{t+h} + \Delta\mathrm{t}, \mathrm{J} = \mathrm{B}|\mathrm{X(t+h)})$

$$= \sum_{k=1}^{2}\sum_{l=1}^{2} h_B\big(t+h; x(t+h), v_B^l\big) * \exp(-H_D\big(t+h; X(t+h), v_D^k\big))$$
$$* \exp(-H_B\big(t+h; X(t+h), v_B^l\big))\, p_{kl}\Delta t$$

and $C = A + B = \Pr(t + h \le T < t + h + \Delta t | X(t + h))$

$$= \sum_{k=1}^{2}\sum_{l=1}^{2}(h_D\big(t + h; x(t + h), v_D^k\big) + h_B\big(t + h; x(t + h), v_B^l\big)) * \exp(-H_D\big(t + h; X(t + h), v_D^k\big)) * \exp(-H_B\big(t + h; X(t + h), v_B^l\big))\, p_{kl}\Delta t$$

$H_D(.)$ and $H_B(.)$ are the integrals of $h_D(.)$ and $h_B(.)$, respectively.

Tables 9, 10, and 11 present the estimated probabilities for h=1, 3, 6, and 9 months. Tables 9 and 10 show that, given the characteristics of account (1), the credit card issuer should expect such an account to have a very low probability of charge-off by each future date. For instance, as shown in Table 10, the probability of charge-off by age 9 months is 0.98%, by age 11 months is 3.15%, by age 14 months is 6.81%, and by age 17 months is 10.83%. This low probability of charge-off is in agreement with the actual status of account (1) which indicates that the account is active.

Table 9: Probability of Charge-off by Age t+h, Given the Covariates

h (in months)	Account (1)	Account (2)	Account (3)
1	5.27%	59.93%	79.17%
3	7.34%	61.57%	79.84%
6	10.85%	63.85%	80.75%
9	14.69%	65.90%	81.56%

Notes: For the three accounts, the table provides the estimated probabilities of charge-off by age $t+h$, given the covariates. These probabilities are given as $Pr(T \le t + h | X(t + h))$, where $t+h = 9, 11, 14, 17$ months for account (1), $t+h = 51, 53, 56, 59$ months for account (2), and $t+h = 71, 73, 76, 79$ months for account (3).

Table 10: Probability of Charge-off by Age t+h, Given the Covariates and Current Age t

h (in months)	Account (1)	Account (2)	Account (3)
1	0.98%	2.11%	1.67%
3	3.15%	6.13%	4.82%
6	6.81%	11.68%	9.11%
9	10.83%	16.70%	12.94%

Notes: For the three accounts, the table provides the estimated probabilities of charge-off by age t + h, given the covariates and that the account was not charged-off until age t. These probabilities are given as $\Pr(T \le t + h | T > t, X(t + h))$, where t = 8 months and t + h = 9, 11, 14, 17 months for account (1), t = 50 months and t+h = 51, 53, 56, 59 months for account (2), and t = 70 and t + h = 71, 73, 76, 79 months for account (3).

Table 11 shows that if account (1) is ever charged-off at a given future date, it is more likely due to delinquency and not bankruptcy, given its characteristics. In contrast, Tables 9 and 10 present that for accounts (2) and (3), the credit card issuer should expect that both accounts have a high probability of charge-off by each future date, given their characteristics. Moreover, as shown in Table 11, if accounts and (3) are charged-off at a given future date, the chance of delinquency is higher for account (2) and the chance of bankruptcy is higher for account (3), given their characteristics. These findings are also confirmed by the actual status of accounts (2) and (3), which indicates that accounts (2) and (3) are delinquent and bankrupt, respectively.

Table 11: Probabilities of Charge-off due to Delinquency and Bankruptcy, Given the Covariates and Charge-off in the Small Age Interval [t+h ≤ T < t+h+∆t)

		Due to (D)	
h (in months)	Account (1)	Account (2)	Account (3)
1	86.60%	60.60%	49.59%
3	85.04%	59.54%	49.05%
6	82.85%	58.03%	48.34%
9	80.76%	56.63%	47.76%
		Due to (B)	
h (in months)	Account (1)	Account (2)	Account (3)
1	13.40%	39.40%	50.41%
3	14.96%	40.46%	50.95%
6	17.15%	41.97%	51.66%
9	19.24%	43.37%	52.24%

Notes: For the three accounts, the table provides the estimated probabilities of charge-off due to delinquency and the estimated probabilities of charge-off due to bankruptcy, given the covariates and that the account is charged-off in the small age interval $[t + h \leq T < t + h + \Delta t)$. These probabilities are given as $Pr(J = j \mid t + h \leq T < t + h + \Delta t, X(t + h))$, where $j = D, B$, $t+h = 9, 11, 14, 17$ months for account (1), $t + h = 51, 53, 56, 59$ months for account (2), and $t + h = 71, 73, 76, 79$ months for account (3).

7. Conclusion

In this paper, we specify a competing risks model in order to estimate the probability of charge-off through estimating the specific hazard functions for charge-off type (D) and for charge-off type (B). We further allow the specific hazard functions for charge-off type (D) and for charge-off type (B) to depend on cause specific and possibly correlated unobserved heterogeneities. This is the first paper that accounts for delinquency and bankruptcy simultaneously, controls for unobserved covariates, and allows for dependency between the two cause specific hazard functions.

We find a significant negative correlation between the unobserved heterogeneity term v_D and the unobserved heterogeneity term v_B. So, we confirm that the specific hazard functions for charge-off type (D) and for charge-off type (B) are negatively dependent. We also find that there are differences in the impacts of the covariates depending upon the charge-off type. In particular, we show that Behavioral Score, FICO Score, Payment, Balance, Unemployment Rate, and Credit Card Act have a significant impact on the specific hazard function for charge-off type (D); however, only FICO Score, Balance, and Unemployment Rate have a significant impact on the specific hazard function for charge-off type (B).

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