A Subordinated Stochastic Framework for Supervisory Stress Testing

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In this study we develop and demonstrate a combined stochastic framework for supervisory stress tests that assesses the probable first passage time and the time-related likelihoods for banks to breach their regulatory minimum capital ratios. Our proposed framework allows regulators to intuitively integrate credit characteristics of the individual loans and risky assets within a bank's portfolio with the idiosyncratic bank's merits (such as the general bank's policies, risk tolerance, size, connectivity, and interconnectedness within the entire banking system). We develop the necessary derivations, illustrate the stochasticity of the measurements through several Monte Carlo simulations, and further draw inferences from some sensitivity analyses for the model's parameters. The proposed stress testing framework can assist monitored financial institutions, policy makers, and regulatory bodies.

JEL classification: C30, E58, G21
Keywords: Supervisory Stress Tests; Banking System; Subordinated Stochastic Process; Inverse Gaussian distribution; First Hitting Time

1. Introduction

In this study we develop a subordinated stochastic model for supervisory banks' stress tests. We present a simple and highly intuitive approach to assess the likely capital deterioration of banks in light of hypothetical adverse economic scenarios. Our proposed framework hereafter allows regulatory bodies and policymakers to consider both the credit characteristics of the individual assets within a bank's portfolio and the bank's overall attributes. Both aspects logically impact banks' survivability. Our model takes a more holistic attitude than other related frameworks and thus may improve the quality of supervisory stress tests.

Our framework contains four steps. First, each loan and risky asset within the bank's portfolio is analyzed based on its exposure time and possibly-changing exposure or declining rates. Second, we accumulate the expected singular capital losses in every time unit (quarter of a year, for instance) and compute the naïve drift and diffusion of the entire bank's capital losses. Third, we integrate vectors of idiosyncratic covariates based on the inspected bank's specific characteristics and obtain both the forward-looking mean and the variance rates of the inclusive bank's capital loss over time. Fourth, using the properties of the Inverse Gaussian distribution, we process the first passage time and the projected time to bank failure, when the bank's excess capital beyond the minimum regulatory threshold is completely depleted. We also calculate the likelihoods for bank failure at diverse phases in the future and deploy several stochastic Monte Carlo simulations to illustrate the probable paths the bank's surplus capital progresses through.

Supervisory stress tests simulate how well individual banks could withstand theoretical future macroeconomic scenarios. These adverse scripts are not projected. They only incorporate extreme yet feasible economic shocks. For instance, the Dodd-Frank Wall Street Reform and Consumer Protection Act (signed into Federal law on July, 21, 2010) orders the Federal Reserve to alternate trajectories for 28 macroeconomic variables along three general settings (baseline, adverse, and severely adverse).[1] The Federal Reserve then simulates how these economic shocks would affect individual banks' balance sheets, risk-weighted-assets, income statements, capital levels, and four regulatory capital ratios (including common equity tier one ratio, tier one risk-based capital ratio, total risk-based capital ratio, and tier one leverage ratio) over a nine-quarter standard "planning horizon."

Within these supervisory bank stress tests, each of the three simulated economic scenarios (baseline, adverse, and severely adverse) contains certain assumed paths for the 28 macro variables along the planning horizon, where the degrees of economic shocks are arbitrarily selected.[2] For example, in the Board of Governors of the Federal Reserve System (2016a), the severely adverse scenario depicts an increase of 5% by the middle of 2017 (to a level of 10%) in the unemployment rate, where the consumer price inflation rises from about 0.25% in the first quarter of 2016 to an annual rate of 1.25% by the end of the recession. In this severely adverse scenario, equity prices are assumed to fall approximately 50% through the end of 2016, housing prices to drop 25% by the third quarter of 2018, and commercial real estate prices to fall 30% through the second quarter of 2018. Nonetheless, the adverse scenario describes an increase of 2.5% by the middle of 2017 (to a level of 7.5%) in the unemployment rate, where the consumer price inflation rises to an annual rate of 1.75% by the first quarter of 2019. In this adverse scenario equity prices are assumed to fall roughly 25% through the end of 2016, housing prices to drop 12% by the third quarter of 2018, and commercial real estate prices to fall 12% through the third quarter of 2017.

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[1] These 28 variables include 16 general variables in three categories, as follows. The first category consists of U.S. economic activity measures (real Gross Domestic Product (GDP) growth, nominal GDP growth, real disposable income growth, nominal disposable income growth, unemployment rate, and Consumer Price Index (CPI) inflation rate). The second category consists of assets prices in the main financial markets (Dow Jones total stock market index, house price index, commercial real estate price index, and market volatility index). The third category consists of various interest rates (3-month Treasury rate, 5-year Treasury yield, 10-year Treasury yield, BBB corporate yield, mortgage rate, and prime rate). An additional 12 variables assess the real GDP growth, the inflation rate, and the U.S./foreign currency exchange rate in each of the following four international markets: the Euro zone, Asia, Japan, and the U.K.

[2] According to the Board of Governors of the Federal Reserve System (2016a) the approach to model their stress tests reflects "an independent supervisory perspective," and the designated economic trajectories are "forward-looking and may incorporate outcomes outside of historical experience."

Upon a completion of the supervisory stress tests, the Federal Reserve conducts its annual Comprehensive Capital Analysis and Review (CCAR) program. This is a fairly new regulatory agenda that considers both qualitative and quantitative measures.[3] It assesses, regulates, and monitors banks in terms of their overall capital policies and adequacy. Within this program, regulators can approve or reject future planned capital distributions, such as dividends, share repurchases, or executive bonuses, and under some circumstances, may restrict firm-wide practices (e.g. lending to specific segments of the market) or even recommend emergency capital injections to the banks by the Federal Reserve.[4] Naturally, because of possible regulatory intervention, the results of the supervisory stress tests could be substantially different pre-CCAR and post-CCAR. The CCAR program aims to assure that each bank "maintains post-stress capital ratios that are above the applicable minimum regulatory capital ratios in effect during each quarter of the planning horizon."[5]

Our approach for supervisory stress testing is different from prior approaches (as summarized hereafter) by first assessing the likely capital losses incurred by the underlying bank's loans and risky assets from the adverse economic scenario tested along the planning horizon. This task is done by combining the exposure times and the possibly-transforming exposure or depreciation rates for each risky asset. Second, we integrate the implied stochastic behavior of the aggregated capital deterioration for the entire tested bank with its idiosyncratic determinants, which, depending on the context, can be inferred by inspecting the entire banking system or particular segments of the market. Our proposed framework allows regulators and policymakers to combine credit characteristics of the individual assets within a bank's portfolio with the bank's overall attributes. Our model is both intuitive and pragmatic thus conceivably can improve the quality of supervisory stress tests. The present contribution is enhanced in light of the recent (2008-2009) financial crisis across the U.S. banking system.

The study proceeds as follows. In Section 2 we provide a brief economic literature review on supervisory bank stress tests. In Section 3 we assemble our proposed framework. In Section 4 we illustrate the present model with some notional yet realistic Monte Carlo simulations and further examine several related sensitivity analyses. In Section 5 we conclude and point to future related lines of research.

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[3] The CCAR program was first launched in 2011 while deployed primarily for sensitivity analyses. It replaced the Supervisory Capital Assessment Program (SCAP) from 2009. Only in 2012 was the CCAR used as one of the key inputs in decision making for the Federal Reserve and hence disclosed publicly.
[4] Institutions that require emergency capital injections conventionally enter into special commitments to issue convertible preferred securities to the U.S. Treasury. These institutions will get temporary permissions (up to six months) to raise private capitals in public markets to meet their regulatory required minimum capitals and would be able to abandon their commitments without any penalties.
[5] Interested readers can find more information on the recent CCAR program in the publication of the Board of Governors of the Federal Reserve System (2016b).

2. Related Literature

Various scholars advise employing the Value at Risk (VaR) and the Extreme Value Theory (EVT) methodologies as the principal framework for stress testing. Dimson and Marsh (1997) use the worst outcome of a portfolio value to compute the risk of a specific position. Jackson, Maude, and Perraudin (1997) examine the empirical performance of different VaR models using data on the actual fixed income, foreign exchange, and equity security holdings of a large bank. Kupiec (1998, 1999) offers generic methodologies that parameterize stress tests' scenarios with the conditional probability distributions typically used in the VaR applications. Longin (2000) suggests that stress tests should extract the limiting distributions of extreme value computations (minimum and maximum return observations over a given time-period). Tan and Chan (2003) elaborate on these methods and examine whether the underlying assumption of normality is adequate for stress testing under the StressVaR and the StressVaR-x procedures.

Bangia, Diebold, Kronimus, Schagen, and Schuermann (2002) link credit migration matrices to contractionary and expansionary business cycles as a conceptual framework for stress testing credit portfolios. Alexander and Sheedy (2008) incorporate both volatility clustering and heavy tails and evaluate the performance of eight market risk models. Berkowitz (1999) proposes integrating economic scenarios with their corresponding probabilities, which can be inferred from past events. Aragones, Blanco, and Dowd (2001) integrate bank stress tests into a formal market risk model. Jacobs, Karagozoglu, and Sensenbrenner (2015) further utilize a Bayesian approach to stress test credit risk portfolios. Kopeliovich, Novosyolov, Satchkov, and Schachter (2015) take an inverse approach and show how to select the most likely stress test scenario that generates a specific capital loss.

Other inquiries stress test retail loan portfolios in particular. Kearns (2004) documents this application over Irish retail credit institutions and finds some evidence that the level of loan losses, judged by loan-loss provisions, rises when GDP growth declines, but even more significantly when unemployment escalates. Rösch and Scheule (2007) develop a framework to stress test the smallest building blocks of a portfolio, i.e. the loans themselves, while accounting for the cross-correlations in each portfolio. Breeden, Thomas, and McDonald (2008) further present a vintage approach for stress testing retail loan portfolios, which is based on the common origination time for the specific loans within. Drehmann, Sorensen, and Stringa (2010) measure credit and interest rate risks jointly and show how together they affect banks' economic values and capital adequacies. Fender, Gibson, and Mosser (2001), Foglia (2009), and Jacobs (2013) provide surveys on various approaches for bank stress testing.

A few recent studies examine the interactions among economic shocks in banks' stress tests. Numpacharoen (2013) demonstrates how to utilize a correlation matrix for the purpose of stress testing. Parnes and Jacobs (2018) present a differential equations model for supervisory stress testing that considers the probable correlations among macroeconomic shocks and predicts the likely impact of possible regulatory intervention at the CCAR stage.

3. The Model

3.1 The Subordinated Stochastic Framework

To construct a universal framework for supervisory stress tests of financial institutions we consider a common depository bank that has multiple loans outstanding and risky assets in its current portfolio. The values of these assets are likely to deteriorate during an adverse economic scenario although at different rates. In addition, the bank may hold these risky assets throughout the entire planning horizon, but it can also terminate some of these holdings before the end of the projected period. We therefore assign $\{I_j(\tau)\}$ to be the following indicator process:

$$I_j(\tau) = \begin{cases} 1 & \text{if bank loan or risky asset j exists during contractionary time } \tau \\ 0 & \text{otherwise} \end{cases}. \quad (1)$$

In this framework, the stochastic process $\{\Delta(\tau)\}$, representing the expected cumulative capital deterioration in the underlying bank's holdings at adverse time τ, becomes:

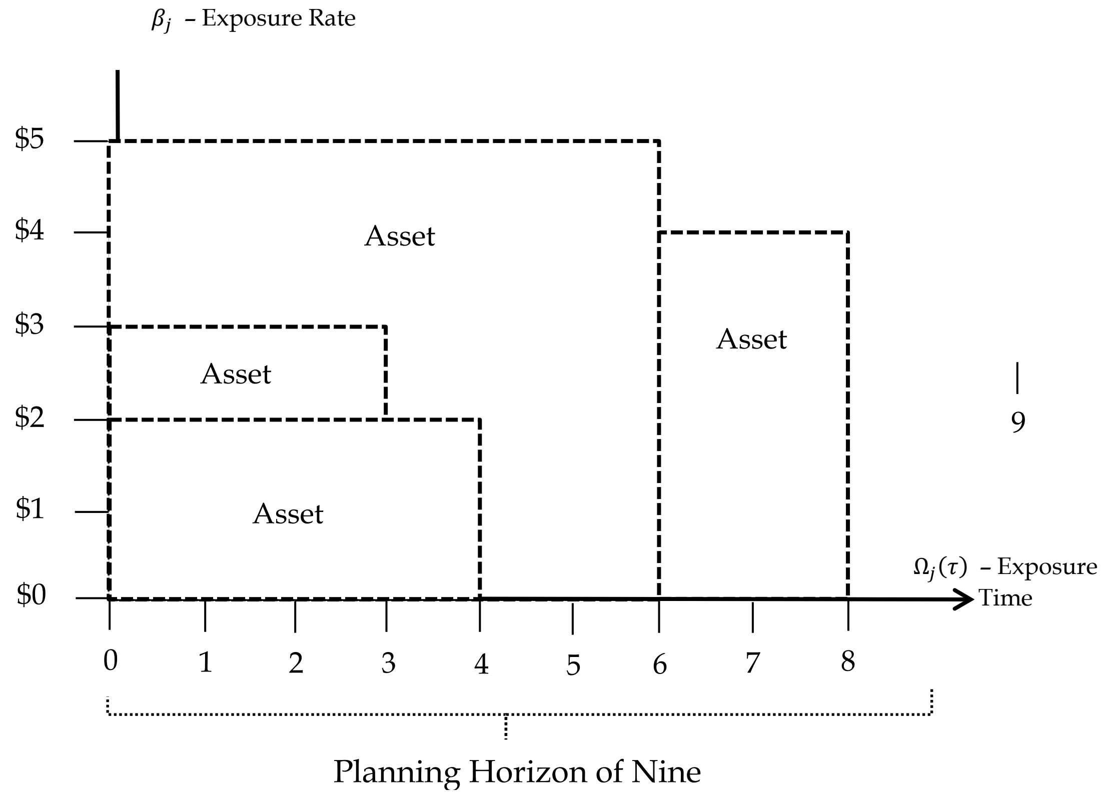
$$\Delta(\tau) = \sum_{j=1}^{N} \int_0^{\tau} \beta_j I_j(u) du = \sum_{j=1}^{N} \beta_j \Omega_j(\tau), \quad (2)$$

where N is the total number of bank loans or risky assets, β_j denotes the rate at which the value of risky asset j deteriorates (and as a result the pace at which the bank's capital that is strictly associated with risky asset j declines), and $\Omega_j(\tau)$ signifies the inclusive time that loan or risky asset j remains in the bank's portfolio during the adverse economic circumstances examined. $\Omega_j(\tau)$ is typically observable, since each bank loan has its own expected maturity date, thus regulators can contrast this period with the planning horizon of the specific economic scenario. $\beta_j, j = 1,..,N$, on the other hand, is merely quantifiable from the Federal Reserve proprietary internal models and the severity assigned to the specific economic scenario tested.

In essence, the expected cumulative capital deterioration $\Delta(\tau)$ in the underlying bank's holdings at adverse time τ is the sum of the products between exposure times $\Omega_j(\tau)$ and exposure or deterioration rates β_j. This summation is conducted over all the risky assets and bank loans $j = 1,..,N$ within the bank's portfolio throughout the planning horizon of the stress test. To enhance intuition, the anticipated cumulative capital deterioration $\Delta(\tau)$ can be graphically captured by aggregating all the overlapping areas formed by placing the exposure rates β_j along the vertical axis and the respective exposure times $\Omega_j(\tau)$ along the horizontal axis for the diverse bank loans and risky assets, as illustrated in **Figure I**.

We should note here that the rate β_j at which the value of risky asset j deteriorates does not have to be constant all along its exposure time. This rate may change according to the dynamics of the economic scenario tested. For instance, a severe economic shock that subsides at a later stage may cause the value of a certain risky asset to drop at a rapid pace at first, but then the rate of deterioration in its value should gradually decelerate. Exposure rates can further exhibit upward or downward sloping paths. They can also form either linear or nonlinear curves. In all of these cases regulators should combine the disjoint areas (or integrals) shaped by the alternating exposure worsening rates and exposure time intervals.

Figure I : An Illustration of the Expected Cumulative Capital Deterioration $\boldsymbol{\Delta(\tau)}$



Above is a simple graphical illustration of the assessment of the bank's expected cumulative capital deterioration as captured in equation (2): $\Delta(\tau) = \sum_{j=1}^{N} \beta_j \Omega_j(\tau)$. Based on the Federal Reserve proprietary models and the specific adverse economic scenario tested, risky asset #1 is expected to deteriorate at a pace of $2,000 per quarter over the next four quarters. Risky asset #2 is projected to lose $3,000 every quarter over the next three quarters. Risky asset #3 is expected to decline at a rate of $5,000 each quarter in the next six quarters. The value of risky asset #4 is anticipated to drop at a pace of $4,000 every quarter throughout the next eight quarters (close to the standard planning horizon of nine quarters in the supervisory stress tests). In this case the entire bank's portfolio is forecasted to have a cumulative capital deterioration of: $\Delta(\tau) = (\$2K \times 4) + (\$3K \times 3) + (\$5K \times 6) + (\$4K \times 8) = \$79K$.

Nevertheless, the cumulative capital deterioration $\Delta(\tau)$ is only projected based on a given economic scenario as set by the hypothetical supervisory stress test. It is neither fixed nor precisely forecasted. Its actual trajectory is subject to unsettled economic forces as elaborated later on. Since the cumulative capital deterioration $\Delta(\tau)$ is influenced by varying economic determinants, the enclosing stochastic process $\{\Psi[\Delta]\}$ describes the collection of likely paths of the bank's total surplus capital beyond the minimum regulatory level. Naturally, each of these stochastic processes is a function of the current bank's excess capital Ψ_0 beyond the minimum regulatory threshold, the mean rate (the drift) μ_Ψ and the variability (the diffusion) σ_Ψ^2 of the bank's cumulative capital loss per unit time. The combined stochastic processes $\{\Psi[\Delta(\tau)]\}$ is therefore called a "subordinated process" where $\Delta(\tau)$ is named the "directing process" and $\Psi[\Delta]$ is titled the "parent process."

3.2 The Overlay First Hitting Time Module

Regulators would normally classify a failed bank whenever its capital falls below the minimum regulatory capital ratio (equity that must be held as a percentage of risk-weighted assets). We can easily scale the dollar amount buffer between the current bank's capital and the minimum regulatory threshold to different intervals. For convenience, we shall map this buffer so Ψ_0 is assigned to express the current excess capital held, while level zero conveys the cutoff point between an operational or an active bank and a failed bank. Therefore, we can portray the First Hitting Time (FHT) $\alpha = Min\{\tau : \Psi[\Delta(\tau)] \leq 0\}$ for a bank to fail as a Wiener diffusion process with random variation (bidirectional movements).[6] This process trails the inverse Gaussian distribution with the following Probability Density Function (PDF):[7]

$$f(\alpha|\Psi_0, \mu_\Psi, \sigma_\Psi^2) = \frac{\Psi_0}{\sqrt{2\pi\alpha^3\sigma_\Psi^2}} exp\left[-\frac{(\Psi_0 + \alpha\mu_\Psi)^2}{2\alpha\sigma_\Psi^2}\right]. \tag{3}$$

The survival function of α, exemplifying the likelihood that a bank would not fail hence would remain operational at least until time α, attains the complementary of the Cumulative Distribution Function (CDF). From the properties of the inverse Gaussian distribution it is:

$$F^C(\alpha|\Psi_0, \mu_\Psi, \sigma_\Psi^2) = \Phi\left[\frac{\alpha\mu_\Psi + \Psi_0}{\sqrt{\alpha\sigma_\Psi^2}}\right] - exp\left(\frac{-2\Psi_0\mu_\Psi}{\sigma_\Psi^2}\right)\Phi\left[\frac{\alpha\mu_\Psi - \Psi_0}{\sqrt{\alpha\sigma_\Psi^2}}\right], \tag{4}$$

where $\Phi[\ \ ]$ denotes the CDF of the standard Normal distribution. The outcome is a censored survival time α out of the planning horizon. The ergodic properties of this Wiener process are well documented in the mathematical literature. Whenever $\mu_\Psi < 0$ the zero threshold will be breached with probability 1, hence the examined bank will continue to drift towards zero excess capital, and eventually it will fail regardless of the variability of the stochastic process.[8] The same ergodic property arises when $\mu_\Psi = 0$, although in this case the inevitable bank failure is a direct result

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[6] We explicitly show how to generate the stochastic process $\Psi[\Delta(\tau)]$ in the later section of Monte Carlo Simulations, while also embedding the exposure rates β_j and the exposure times $\Omega_j(\tau)$.

[7] The inverse Gaussian distribution has become a popular dissemination to depict first passage time situations of Brownian motions in numerous disciplines ever since 1915. Seshadri (1998) elaborates on this distribution, its advances, and its broad applications.

[8] The ergodic properties clearly aim towards a distant horizon, yet in our context of supervisory stress tests the behavior of this Wiener process is examined along a planning horizon of nine quarters of a year.

of the variability itself. When $\mu_\Psi > 0$, however, the stochastic process tends to drift away from the failure threshold set at zero excess capital, thus a bank failure is not certain any more.

If we let BF to represent the event of *eventual* Bank Failure (captured over a very long time frame) as an explicit outcome of the adverse economic scenario tested, i.e. when other possible competing risks are ignored, then the probability of process absorption is attained as:

$$P(BF) = \lim_{\alpha\to\infty}[1 - F^C(\alpha|\Psi_0, \mu_\Psi, \sigma_\Psi^2)] = \begin{cases} 1 & \text{for } \mu_\Psi \le 0 \\ exp\left(\frac{-2\Psi_0\mu_\Psi}{\sigma_\Psi^2}\right) & \text{for } \mu_\Psi > 0 \end{cases}. \quad (5)$$

This fairly compact result evolves because we allow the FHT $\alpha = Min\{\tau: \Psi[\Delta(\tau)] \le 0\}$ to continue increasing, i.e. we drive α to infinity, then both the CDFs of the standard Normal distribution in equation (4) converge to 1. Equation (5) is also rational since if we scale the variance parameter to unity, i.e. when we scale the process to have $\sigma_\Psi^2 = 1$, then as long as the drift is strictly positive ($\mu_\Psi > 0$), the likelihood of bank failure depends only on the product $\Psi_0\mu_\Psi$. The larger this product, the lower the risk of a bank failure from the contractionary economic cycle tested and vice versa. Essentially, the bigger the current bank's excess capital Ψ_0 and/or the stronger the positive drift μ_Ψ away from the zero failure threshold, the better the chances the examined bank has to survive the adverse economic scenario, and vice versa.

Clearly, the likelihood for a bank failure within a specified time period ξ is the complementary to equation (4) hence:

$$P(BF|\alpha = \xi) = 1 - F^C(\xi|\Psi_0, \mu_\Psi, \sigma_\Psi^2) = 1 - \Phi\left[\frac{\xi\mu_\Psi + \Psi_0}{\sqrt{\xi\sigma_\Psi^2}}\right] + exp\left(\frac{-2\Psi_0\mu_\Psi}{\sigma_\Psi^2}\right)\Phi\left[\frac{\xi\mu_\Psi - \Psi_0}{\sqrt{\xi\sigma_\Psi^2}}\right] \quad (6)$$

Conditional on the event BF the mean survival time of the bank is further given by:

$$E[\alpha|BF] = \frac{\Psi_0}{|\mu_\Psi|} \quad \text{for } \mu_\Psi \neq 0. \quad (7)$$

This expected survival time, also representing the mean time to process absorption (i.e. to a bank failure) is highly intuitive. For example, when the current bank's excess capital beyond the minimum regulatory threshold is set as $\Psi_0 = \$10\text{M}$, but the continuous mean rate of capital deterioration (the drift) is $\mu_\Psi = -\$1\text{M}$, then on average it should take the examined bank 10 time units to reach the zero excess capital threshold and fail.

3.3 *The Underlying Economic Forces*

As mentioned earlier, the cumulative capital deterioration $\Delta(\tau)$ is subject to various economic forces that make it stochastic rather than fixed. These economic forces affect the likely paths of the bank's inclusive capital and they normally include both macroeconomic and idiosyncratic covariates. The macroeconomic variables that shape $\Delta(\tau)$ can be largely summarized by the supervisory stress test's 28 macro variables.

The idiosyncratic determinants, however, consist of the specific geographic location at which the examined bank operates, the unique hazards for the underlying bank (such as excess exposure to subprime mortgages or a biased portfolio towards the energy sector, for instance), the particular policies and the general attitude of the bank's management towards risk taking, the size and the relative connectivity of the bank within the entire banking system, the interconnectedness of the bank loans and the risky assets themselves, etc. These idiosyncratic variables should be profoundly analyzed at the CCAR stage, where comparative analysis of different banks in the system is within reach.

The 28 macroeconomic determinants set by the adverse scenario of the stress test have already dictated the likely rate β_j of capital losses per risky asset per unit time. However, the idiosyncratic determinants influence the drift μ_Ψ and the diffusion σ_Ψ^2 of the stochastic process underlying the bank's capital deterioration. While the current bank's excess capital Ψ_0 beyond the minimum regulatory threshold is observable, both the mean μ_Ψ and the variance σ_Ψ^2 of the directing stochastic process $\Delta(\tau)$ are computable from their likely tracks throughout the planning horizon, though further premiums or discounts to these figures should be taken according to the idiosyncratic determinants with a comparative analysis to other examined banks as follows:

$$\begin{cases} \mu_\Psi = \widetilde{\mu_\Psi} + \mathfrak{J}\lambda \\ \sigma_\Psi^2 = \widetilde{\sigma_\Psi^2} + \mathfrak{J}\omega' \end{cases} \tag{8}$$

where $\widetilde{\mu_\Psi}$ and $\widetilde{\sigma_\Psi^2}$ are the notionally assessed (naïve) drift and diffusion of $\Delta(\tau)$, $\mathfrak{J}$ denotes a row vector of idiosyncratic covariates, and λ and ω are two column vectors of coefficients for the mean and the variance of $\Delta(\tau)$, respectively. Depending on the regulatory needs, these coefficients can be assessed by studying the entire banking system or explicit segments within.

4. Monte Carlo Simulations

We now demonstrate the computations of the model parameters, deploy several Monte Carlo simulations that display the overall stochasticity of a bank's excess capital beyond the minimum regulatory threshold under a given stress test, and further explore some sensitivity analyses with respect to the underlying economic determinants.

In Panel A of **Table 1** we present a generic heterogeneous bank portfolio that contains ten risky assets with diverse behavior patterns. Seven of these risky assets have exposure times $\Omega_j(\tau)$ that stretch throughout the entire standard planning horizon of nine quarters. However, three holdings (asset #2, asset #8, and asset #10) are more limited and have shorter exposure times. Two of the risky assets (asset #2 and asset #6) exhibit fixed exposure rates β_j, while the others are projected to have varying capital deterioration rates over time. Asset #9 and asset #10, for example, have monotonically decreasing exposure rates. The expected exposure rates of asset #7 are almost entirely escalating. The exposure rates of asset #1 gradually (non-monotonically) decay, while asset #3, asset #5, and asset #8 experience cycles of different magnitudes and lengths in their exposure rates. In the last column of Panel A we summarize $\Delta_j(\tau) = \beta_j \Omega_j(\tau)$ for the individual risky assets and $\Delta(\tau) = \sum_{j=1}^{N} \beta_j \Omega_j(\tau) = \$7M$ in equation (2) for the aggregated bank portfolio during the complete planning horizon. At the bottom of Panel A we further sum the projected capital losses per quarter. These figures assist us in computing the naïve drift and diffusion of the subordinated stochastic process $\{\Psi[\Delta(\tau)]\}$.

In panel B we first state the bank's current excess capital beyond the minimum regulatory level as $\Psi_0 = \$6M$ (arbitrarily nominated below the forecasted capital loss $\Delta(\tau) = \$7M$ to realistically engage bank failure). We then compute both the naïve average capital loss per quarter as $\widetilde{\mu_\Psi} = -\$777.78K$ (in our context, a negative drift represents capital deterioration, where a positive drift signifies capital accumulation) and a raw variance of $\widetilde{\sigma_\Psi^2} = \$47{,}377.95K$. At this stage we do not add any further input from the underlying economic determinants (we shall explore their corresponding effects in the later sensitivity analyses), hence we simply set $\mathfrak{J}\lambda = 0$ and $\mathfrak{J}\omega = 0$ in equation (8), so $\mu_\Psi = \widetilde{\mu_\Psi} = -\$777.78K$ and $\sigma_\Psi^2 = \widetilde{\sigma_\Psi^2} = \$47{,}377.95K$.

In Panel C we conduct the model's computations and study the bank's likelihoods of failure along the nine quarters that assemble the standard planning horizon. The first column situates possible FHT as $\alpha = 1, 2, \ldots, 9$ consecutive quarters of a year. The second column assesses the gradually declining survival chances as captured in equation (4). The third column simply reveals from equation (5) that, if nothing changes over the long run, the examined bank is doomed to fail because of its negative drift for excess capital. The fourth column processes the probabilities for bank failure (in light of the adverse economic scenario tested) given different failure times from equation (6). As detected, the likelihoods progressively grow from negligible values at the beginning to 94.3% after nine quarters.

It is worth mentioning here that although the initial bank's surplus capital above the minimum required by law is set as $\Psi_0 = \$6M$, and the bank is projected to lose $\Delta(\tau) = \$7M$ over the next nine quarters (16.667% more than Ψ_0), a bank failure is not guaranteed within this time interval. This uncertainty evolves due to the stochasticity of the subordinated process as observed by $\sigma_\Psi^2 > 0$. In the fifth column we then compute the expected time to bank failure from equation (7) and obtain $E[\alpha|BF] = 7.714$ quarters of a year. This figure should serve policymakers as a rough estimation for the most probable time that the tested bank could fail.

In **Figure II** we deploy ten Monte Carlo simulations that represent feasible paths for the bank's excess capital beyond the minimum regulatory threshold. All of the simulations start from the initial bank's excess capital $\Psi_0 = \$6M$ and then follow the random iterative process $\Psi_\tau = \Psi_{\tau-1} exp\left[\frac{\mu_\Psi + \sigma_\Psi \times NORMINV(RAND(\ \ ),0,1)}{\Psi_{\tau-1}}\right]$, where $NORMINV(RAND(\ \ ), 0,1)$ randomly draws numbers from the standard Normal distribution (with a mean of zero and a unit standard deviation). Every new run, this recurring stochastic process generates a different possible path hence the ten-

Table 1
Illustrative Example for the Model

Panel A: Input Data (in $K)

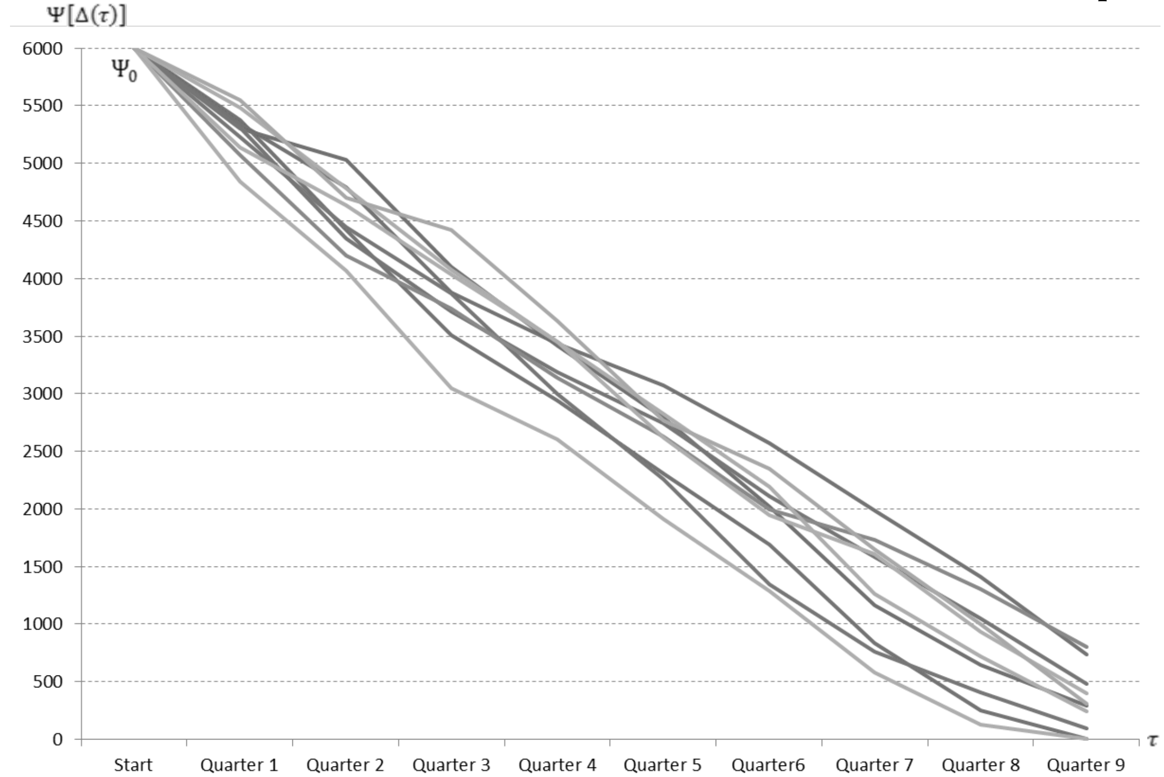
β_j (in $K)	$\Omega_j(\tau)$									$\Delta(\tau)$ (in $K) Eq. (2)
	Quarter 1	Quarter 2	Quarter 3	Quarter 4	Quarter 5	Quarter 6	Quarter 7	Quarter 8	Quarter 9	
Asset #1	85	85	60	60	50	50	50	50	40	530
Asset #2	130	130	130	130	130	130	0	0	0	780
Asset #3	80	90	100	110	100	90	80	70	60	780
Asset #4	100	100	150	150	150	100	50	50	10	860
Asset #5	70	80	90	80	70	80	90	80	70	710
Asset #6	100	100	100	100	100	100	100	100	100	900
Asset #7	50	52	54	56	58	60	62	64	64	520
Asset #8	120	130	140	150	140	130	120	0	0	930
Asset #9	110	100	90	80	70	60	50	40	30	630
Asset #10	80	70	60	50	40	30	20	10	0	360
Sum:	925	937	974	966	908	830	622	464	374	7,000

Table 1 : Continued

Panel B: Model Parameters (in $K)						
Ψ_0	$\widetilde{\mu_\Psi}$	$\widetilde{\sigma^2_\Psi}$	$\Im\lambda$	$\Im\omega$	μ_Ψ	σ^2_Ψ
6,000	-777.78	47,377.95	0	0	-777.78	47,377.95
Arbitrarily Selected					Equation (8)	Equation (8)

Panel C: Output Data				
First Hitting Time α	$F^C(\alpha \vert \Psi_0, \mu_\Psi, \sigma^2_\Psi)$	$P(BF)$	$P(BF \vert \alpha)$	$E[\alpha \vert BF]$
	Equation (4)	Equation (5)	Equation (6)	Equation (7)
1 Quarter	1.000	1	0.000	7.714
2 Quarters	1.000	1	0.000	Quarters
3 Quarters	1.000	1	0.000	
4 Quarters	1.000	1	0.000	
5 Quarters	1.000	1	0.000	
6 Quarters	0.993	1	0.007	
7 Quarters	0.820	1	0.180	
8 Quarters	0.340	1	0.660	
9 Quarters	0.057	1	0.943	

Figure II
Ten Monte Carlo Simulations for the Stochastic Behavior of the Bank's Excess Capital



-presented trails are merely representative.[9] As observed, out of the ten simulations, two paths lead to bank failure. When detected $\Psi_{\tau=9} \cong 0$, the bank's surplus capital is completely eradicated after nine quarters of a year and the bank is no longer considered operative.

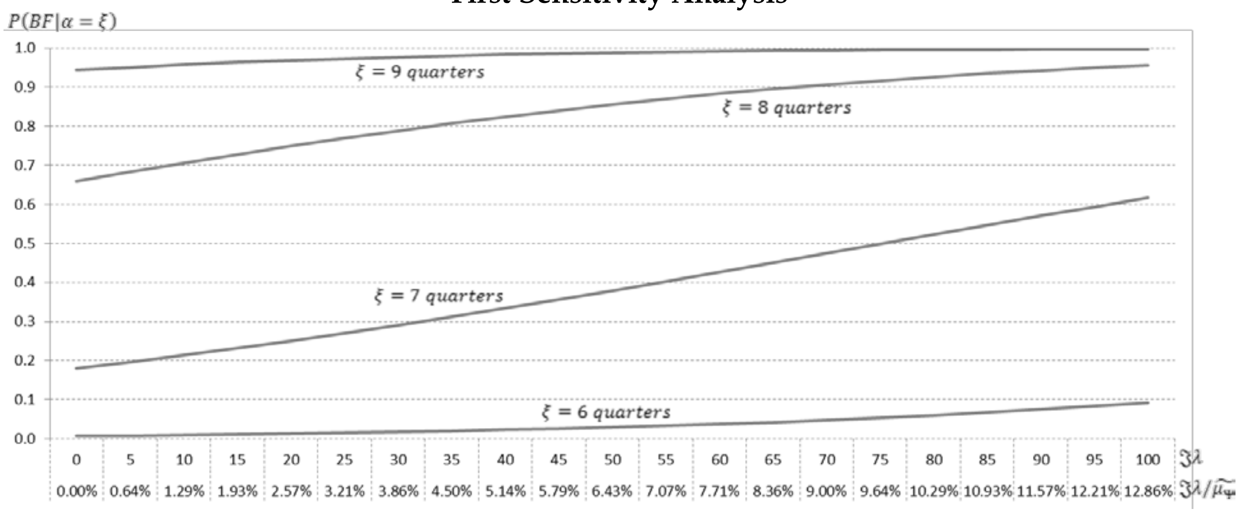
We execute ten Monte Carlo simulations that represent feasible paths for the bank's excess capital beyond the minimum regulatory threshold. All of these simulations start from $\Psi_0 = \$6M$ and follow the random process $\Psi_\tau = \Psi_{\tau-1} exp\left[\frac{\mu_\Psi + \sigma_\Psi \times NORMINV(RAND(\ \ ),0,1)}{\Psi_{\tau-1}}\right]$, where $NORMINV(RAND(\ \ ),0,1)$ randomly draws numbers from the standard Normal distribution (i.e. with a mean of zero and a unit standard deviation). This iterative stochastic process generates every new run a different possible path hence the ten selected trails are merely representative. As observed, out of the ten simulations, two paths lead to an apparent bank failure, where $\Psi_{\tau=9} \cong 0$ indicates that the bank's surplus capital is completely eradicated after nine quarters of a year.

We can now embed further input from the underlying economic determinants as follows. In **Figure III** we test the consequences of confronting diverse magnitudes of the

[9] Clearly, with more simulated stochastic runs, the model achieves greater forecasting accuracy.

idiosyncratic determinants $\Im\lambda$ (placed along the horizontal axis) on the time-dependent failure probability $P(BF|\alpha=\xi)$ (placed along the vertical axis). The idiosyncratic determinants $\Im\lambda$ influence the stochastic process drift μ_Ψ in equation (8), which then affects the bank's failure probability $P(BF|\alpha=\xi)$ in equation (6). We therefore record on the horizontal axis of **Figure III** both $\Im\lambda$ in absolute (thousands of) dollar amounts and their respective proportions out of the naïve drift $\widetilde{\mu_\Psi}=-\$777.78\text{K}$ as captured by the projected capital losses in this stress test.

Figure III
First Sensitivity Analysis



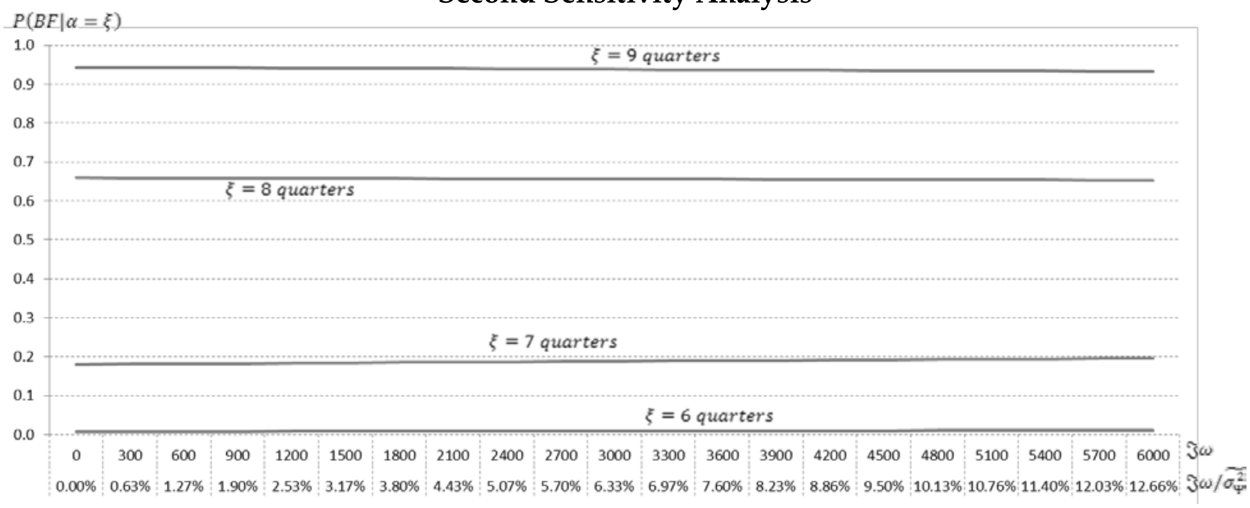
In this sensitivity analysis we examine, ceteris paribus, how bank failure probabilities $P(BF|\alpha=\xi)$ (positioned on the vertical axis) change along different time horizons (6, 7, 8, and 9 quarters of a year) with respect to the process drift μ_Ψ once affected by the idiosyncratic determinants $\Im\lambda$ in equation (8). On the horizontal axis we record both $\Im\lambda$ in absolute (thousands of) dollar amounts and their respective proportions out of the naïve drift $\widetilde{\mu_\Psi}=-\$777.78\text{K}$ as captured by the projected capital losses in the stress test.

As observed, the idiosyncratic economic determinants $\Im\lambda$ (which we recall include the specific geographic location at which the examined bank operates, the specific hazards that the bank is exposed to, the particular policies and the general attitude of the bank towards risk taking, the size and the relative connectivity of the bank within the entire banking system, the interconnectedness of the bank loans and the risky assets themselves) convey positive and meaningful relationship towards the time-dependent bank's failure likelihood $P(BF|\alpha=\xi)$. We further notice that for the six- and the seven-quarter simulations the effects are slightly convex, but for the eight- and the nine-quarter simulations the effects are slightly concave. These phenomena occur in light of the properties of the CDF of the standard Normal distribution in equation (6), which attains an elongated "s" shape with an asymptote of one.

In **Figure IV** we conduct a second sensitivity analysis that assesses how gradual changes in the idiosyncratic determinants $\Im\omega$ affect the stochastic process diffusion σ_Ψ^2

in equation (8) and thus influence the time-dependent failure probability $P(BF|\alpha = \xi)$. As before, to enhance intuition, we record along the horizontal axis of **Figure IV** both $\Im\omega$ in absolute (thousands of) dollar amounts and their respective proportions (comparable percentages to the first sensitivity analysis) out of the naïve diffusion $\sigma_{\Psi}^{2} = \$47{,}377.95\text{K}$ as captured by the projected capital losses in the stress test.

Figure IV
Second Sensitivity Analysis



In this sensitivity analysis we examine, ceteris paribus, how bank failure likelihoods $P(BF|\alpha = \xi)$ (positioned on the vertical axis) change along different time horizons (6, 7, 8, and 9 quarters of a year) with respect to the process diffusion σ_{Ψ}^{2} once affected by the idiosyncratic determinants $\Im\omega$ in equation (8). On the horizontal axis we record both $\Im\omega$ in absolute (thousands of) dollar amounts and their respective proportions out of the naïve diffusion $\sigma_{\Psi}^{2} = \$47{,}377.95\text{K}$ as captured by the projected capital losses in the stress test.

In this experiment, however, we detect a negligible influence of the idiosyncratic determinants $\Im\omega$ on the time-dependent failure likelihood $P(BF|\alpha = \xi)$ across all inspected time spans. This outcome evolves from the fact that the second moment (the stochastic diffusion) exhibits merely an auxiliary impact on the rate at which the bank's excess capital deteriorates. The first moment (the stochastic drift) is understandably the prominent factor shaping the pace at which the bank's excess capital depreciates during the adverse economic cycle as posed by the supervisory stress test. Consequently, policy makers can choose to simplify their inquiries and abandon the added complexity in the computation of the diffusion module in equation (8) (hence to simply assign the naïve variance as the stochastic process diffusion) while still maintaining a fair approximation to the likely paths of the banks' excess capitals.

5. Conclusion

In this study we have developed and demonstrated a subordinated stochastic framework for supervisory bank stress tests to be used by regulatory bodies. Our approach is somewhat different from other frameworks since it allows regulators and policymakers to incorporate credit characteristics of the individual loans and risky assets within a bank's portfolio with the overall bank's idiosyncratic attributes and various macroeconomic variables. Our model is intuitive, parsimonious, and practical and therefore can improve the quality of supervisory stress tests.

Altogether, we have demonstrated that a bank's excess capital beyond the minimum regulatory threshold, which essentially defines whether a bank is considered operational or not in a supervisory stress test, is dictated by two nested stochastic processes. The directing stochastic process is the cumulative capital deterioration. It is derived by the sum (over the entire bank's portfolios) of the products between the exposure times and the erratic deterioration rates of the individual risky assets and loans. The parent stochastic process dominates the bank's excess capital path, and it is influenced by varying economic and idiosyncratic determinants. It is therefore shaped by the consequential drift and diffusion.

By using a notional yet realistic example of a general heterogeneous bank portfolio, we have illustrated how the computations of the specific likelihoods for a bank failure at different time horizons are deployed. We have also exemplified the calculation of the expected (or most likely) time for a bank failure. This projection could be further contrasted with the planning horizon of the underlying supervisory stress test. In addition, we have illuminated the probable paths of the bank's excess capital with several Monte Carlo simulations. Moreover, we have validated the proposed model's sensitivity to diverse magnitudes of the bank's idiosyncratic merits (often qualitative measures though logically influence the future course of the bank).

This combined set of methodologies can be used as decision support tools for regulatory and supervisory agencies when stress testing banks and financial institutions alike. These projections accompanied by their matching probabilities portray together a more comprehensive view on the survivability of a tested bank given an imaginary adverse economic scenario, as opposed to a common binary outcome of either "pass" or "fail" a static supervisory stress test.

For future lines of research we recommend that intrigued scholars contemplate the idiosyncratic determinants that typically affect both the drift and the diffusion of the stochastic process underlying the deterioration of banks' capitals. We further recommend that regulators and policymakers (such as the Office of Financial Research, the Federal Reserve System, or the Office of the Comptroller of the Currency in the U.S.) assemble and continuously update a national database for these bank-specific characteristics.

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